Singular BSDEs & Emissions Derivative Pricing

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- Putting a Price on CO\textsubscript{2} by \textit{internalizing} its Social Cost
- Regulatory Economic Instruments
  - Carbon \textbf{TAX}
  - Permits Allocation & Trading (\textit{Cap-and-Trade})
- Calibrate the Different Schemes for
  - \textbf{MEANINGFUL} \& \textbf{FAIR} comparisons

- \textbf{Dynamic Stochastic General Equilibrium}
- \textbf{Inelastic} Demand (to start with)
  - Electricity Production for the purpose of \textit{illustration}

- Novelty (if any): \textbf{Random} Factors
  - \textbf{Demand} for goods \( \{D^{k}_{t}\}_{t\geq0} \)
  - \textbf{Costs} of Production \( \{C^{i,j,k}_{t}\}_{t\geq0} \)
    - Spot Price of Coal
    - Spot Price of Natural Gas
Emissions (Cap-and-Trade) Markets MAY or MAY NOT exist in the US (and Canada, Australia, Japan, ....)

In any case, a liquid option market ALREADY exists in Europe

- Underlying $\{A_t\}_t$ non-negative martingale with binary terminal value (single phase model)
- Think of $A_t$ as of a binary option
- Underlying of binary option should be Emissions

Need for Formulae (closed or computable)

- Prices and Hedges difficult to compute (only numerically)
- Jumps due to announcements (Cetin et al.)

Reduced Form Models
### Option quotes on Jan. 3, 2008

<table>
<thead>
<tr>
<th>Option Maturity</th>
<th>Option Type</th>
<th>Volume</th>
<th>Strike</th>
<th>Allowance Price</th>
<th>Implied Vol</th>
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**Carmona Singular BSDEs**
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<tr>
<th>Option Maturity</th>
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Allowance price should be of the form

\[ A_t = \lambda \mathbb{E}\{1_N \mid \mathcal{F}_t\} \]

for a non-compliance set \( N \in \mathcal{F}_t \). Choose

\[ N = \{\Gamma_T \geq 1\} \]

for a random variable \( \Gamma_T \) representing the normalized emissions at compliance time. So

\[ A_t = \lambda \mathbb{E}\{1_{\{\Gamma_T \geq 1\}} \mid \mathcal{F}_t\}, \quad t \in [0, T] \]

We choose \( \Gamma_T \) in a parametric family

\[ \Gamma_T = \Gamma_0 \exp \left[ \int_0^T \sigma_s dW_s - \frac{1}{2} \int_0^T \sigma_s^2 ds \right] \]

for some square integrable deterministic function

\[ (0, T) \ni t \mapsto \sigma_t \]
Dynamic Price Model for $a_t = \frac{1}{\lambda} A_t$

- $a_t$ is given by

$$a_t = \Phi \left( \frac{\Phi^{-1}(a_0) \sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}} \right) \quad t \in [0, T)$$

where $\Phi$ is standard normal c.d.f.

- $a_t$ solves the SDE

$$da_t = \Phi'(\Phi^{-1}(a_t)) \sqrt{z_t} dW_t$$

where the positive-valued function $(0, T) \ni t \mapsto z_t$ is given by

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, \quad t \in (0, T)$$
Risk Neutral Densities

Figure: Histograms for each day of a 4 yr compliance period of $10^5$ simulated risk neutral allowance price paths.
Aside: Binary Martingales as Underliers

Allowance prices are given by \( A_t = \lambda a_t \) where \( \{a_t\}_{0 \leq t \leq T} \) satisfies

- \( \{a_t\}_t \) is a martingale
- \( 0 \leq a_t \leq 1 \)
- \( \mathbb{P}\{\lim_{t \to T} a_t = 1\} = 1 - \mathbb{P}\{\lim_{t \to T} a_t = 0\} = p \) for some \( p \in (0, 1) \)

The model

\[
d a_t = \Phi'(\Phi^{-1}(a_t)) \sqrt{z_t} dW_t
\]

suggests looking for martingales \( \{Y_t\}_{0 \leq t < \infty} \) satisfying

- \( 0 \leq Y_t \leq 1 \)
- \( \mathbb{P}\{\lim_{t \to \infty} Y_t = 1\} = 1 - \mathbb{P}\{\lim_{t \to \infty} T_t = 0\} = p \) for some \( p \in (0, 1) \)

and do a time change to get back to the (compliance) interval \([0, T)\)
Feller’s Theory of 1-D Diffusions

Gives conditions for the SDE

\[ da_t = \Theta(a_t) dW_t \]

for \( x \mapsto \Theta(x) \) satisfying
- \( \Theta(x) > 0 \) for \( 0 < x < 1 \)
- \( \Theta(0) = \Theta(1) = 0 \)

to
- Converge to the boundaries 0 and 1
- NOT explode (i.e. NOT reach the boundaries in finite time)

Interestingly enough the solution of

\[ dY_t = \Phi'(\Phi^{-1}(Y_t)) dW_t \]

IS ONE OF THEM!
Explicit Examples

The SDE

\[ dX_t = \sqrt{2}dW_t + X_t\,dt \]

has the solution

\[ X_t = e^t(x_0 + \int_0^t e^{-s}\,dW_s) \]

and

\[ \lim_{t \to \infty} X_t = +\infty \quad \text{on the set } \{\int_0^\infty e^{-s}\,dW_s > -x_0\} \]

\[ \lim_{t \to \infty} X_t = -\infty \quad \text{on the set } \{\int_0^\infty e^{-s}\,dW_s < -x_0\} \]

Moreover, \( \Phi \) is harmonic so if we choose

\[ Y_t = \Phi(X_t) \]

we have a martingale with the desired properties.

Another (explicit) example can be constructed from Ph. Carmona, Petit and Yor on Dufresne formula.
Publicly available option data being unreliable, calibration

**Has to Be Historical !!!!!!**

- Choose **Constant** Market Price of Risk
- **Two-parameter** Family for Time-change

\[
\{ z_t(\alpha, \beta) = \beta(T - t)^{-\alpha} \}_{t \in [0, T]}, \quad \beta > 0, \alpha \geq 1.
\]

Volatility function \( \{ \sigma_t(\alpha, \beta) \}_{t \in (0, T)} \) given by

\[
\sigma_t(\alpha, \beta)^2 = z_t(\alpha, \beta) e^{-\int_0^t z_u(\alpha, \beta) \, du}
\]

\[
= \begin{cases} 
\beta(T - t)^{-\alpha} e^{\beta \frac{T - \alpha + 1 - (T - t)^{-\alpha + 1}}{-\alpha + 1}} & \text{for } \beta > 0, \alpha > 1 \\
\beta(T - t)^{\beta - 1} T^{-\beta} & \text{for } \beta > 0, \alpha = 1
\end{cases}
\]

**Maximum Likelihood**
for $\alpha = 1$, $\beta > 0$, the price of an European call with strike price $K \geq 0$ written on a one-period allowance futures price at time $\tau \in [0, T]$ is given at time $t \in [0, \tau]$ by

$$C_t = e^{-\int_t^\tau r_s ds} \mathbb{E}\{(A_\tau - K)^+ | \mathcal{F}_t\}$$

$$= \int (\lambda \Phi(x) - K)^+ N(\mu_{t,\tau}, \nu_{t,\tau})(dx)$$

where

$$\mu_{t,\tau} = \Phi^{-1}(A_t / \lambda) \sqrt{(T - t)/(T - \tau)} \beta$$

$$\nu_{t,\tau} = \left(\frac{T - t}{T - \tau}\right)^\beta - 1.$$
Price Dependence on $T$ and Sensitivity to $\beta$

Figure: Dependence $\tau \mapsto C_0(\tau)$ of Call prices on maturity $\tau$. Graphs $\square$, $\triangle$, and $\nabla$ correspond to $\beta = 0.5$, $\beta = 0.8$, $\beta = 1.1$. 

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Implied Volatilities $\beta = 1.2$

Implied Volatilities for Different Maturities

- Maturity=1yr
- Maturity=2yr
- Maturity=3yr

Graph showing the relationship between implied volatilities and maturity.
Implied Volatilities $\beta = 0.6, \lambda = 100$

Implied Volatilities for Different Maturities

- Maturity=1yr
- Maturity=2yr
- Maturity=3yr

K

Carmona
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Relax demand inelasticity
Include preferences to relax risk neutrality
(RC-Delarue-Espinosa-Touzi)
"Representative Agent" form already considered in Seifert-Uhrig-Homburg-Wagner, RC-Fehr-Hinz

Mathematical Set-Up (continuous time)
- \((\Omega, \mathcal{F}, \mathbb{P})\) historical probability structure
- \(W\) D-dimensional Wiener process on \((\Omega, \mathcal{F}, \mathbb{P})\)
- \(T > 0\) finite horizon (end of the single compliance period)
- \(\mathcal{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}\) filtration of \(W\)

Goal of equilibrium analysis is to derive pollution permit price \(\{A_t; 0 \leq t \leq T\}\) allowing firms to maximize their expected utilities simultaneously
Emissions Dynamics

Assume allowance price $A = \{A_t; \, 0 \leq t \leq T\}$ exists.

- $A$ is a $\mathbb{F}$-martingale under $\mathbb{Q}$
- $dA_t = Z_t \, dB_t$ for some adapted process $Z$ s.t. $Z_t \neq 0$ a.s. and $B$ is a D-dim Wiener process for spot martingale measure $\mathbb{Q}$
- $A_T = \lambda 1_{[\Lambda, \infty)}(E_T)$ where
  - $\lambda$ is the penalty
  - $E_t = \sum_{i \in I} E^i_t$ is the aggregate of the $E^i_t$ representing the cumulative emission up to time $t$ of firm $i$
  - $\Lambda$ is the cap imposed by the regulator

Assume the following dynamics under $\mathbb{P}$

$$dE^i_t = (b^i_t - \xi^i_t)dt + \sigma^i_t dW_t, \quad E^i_0 = 0.$$

- $\{E^i_t(\xi^i_t \equiv 0)\}_{0 \leq t \leq T}$ cumulative emissions of firm $i$ in BAU
- $\{\xi^i_t\}\}_{0 \leq t \leq T}$ abatement rate of firm $i$
- Assumptions on emission rates $b^i_t$ and volatilities $\sigma^i_t$ to be articulated later
Abatement costs for firm $i$ given by cost function $c^i_t : \mathbb{R} \rightarrow \mathbb{R}$

- $c^i$ is $C^1$ and strictly convex
- $c^i$ satisfies Inada-like conditions for each $t \in [0, T]$

$$
(c^i)'(-\infty) = -\infty \quad \text{and} \quad (c^i)'(+\infty) = +\infty.
$$

- $c^i(0) = \min c^i_t$ ($\xi^i \equiv 0$ corresponds to BAU)

Typical example for $c^i$

$$
\lambda|x|^{1+\alpha},
$$

for some $\lambda > 0$ and $\alpha > 0$.

Each firm chooses its abatement strategy $\xi^i$ and its investment $\theta^i$ in allowances. Its wealth is given by

$$
X^i_t = X^{i,\xi,\theta}_t = x^i + \int_0^T \theta^i_t dA_t - \int_0^T c^i(\xi^i_t) dt - E^i_T A_T.
$$
Preferences of firm $i$ given by a $C^1$, increasing, strictly concave utility function $U^i : \mathbb{R} \rightarrow \mathbb{R}$ satisfying Inada conditions:

$$(U^i)'(-\infty) = +\infty \quad \text{and} \quad (U^i)'(+\infty) = 0.$$ 

The optimization problem for firm $i$ is:

$$V(x^i) := \sup_{(\xi^i, \theta^i) \in A^i} \mathbb{E}^P\{ U^i(X^i_T, \xi^i \theta^i) \}$$

If no non-standard restriction on $A^i$ set of admissible strategies for firm $i$

**Proposition**

If an equilibrium allowance price $\{A_t\}_{0 \leq t \leq T}$ exists, then the optimal abatement strategy $\hat{\xi}^i$ is given by

$$\hat{\xi}^i_t = [(c^i)']^{-1}(A_t).$$

**NB:** The optimal abatement strategy $\hat{\xi}^i$ is independent of the utility function $U^i$!
Finding the Equilibrium Allowance Price

- Recall
  \[
  dE^i_t = \left[ \tilde{b}^i_t - [(c^i)']^{-1}(A_t) \right] dt + \sigma^i_t dB_t, \quad E^i_0 = 0, \text{ for each } i
  \]

- Assume
  - \( \forall i, \tilde{b}^i_t = \tilde{b}^i(t)E^i_t \) or \( \forall i, \tilde{b}^i_t = \tilde{b}^i(t) \)
  - \( \forall i, \sigma^i_t = \sigma^i(t) \).

- Set
  \[
  b := \sum_{i \in I} \tilde{b}^i, \quad \sigma := \sum_{i \in I} \sigma^i, \text{ and } f := \sum_{i \in I} [(c^i)']^{-1}.
  \]

Therefore we have the following FBSDE

\[
\begin{align*}
  dE_t &= \{ b(t, E_t) - f_t(A_t) \} dt + \sigma(t) dB_t, \quad E_0 = 0 \quad (1) \\
  dA_t &= Z_t dB_t, \quad A_T = \lambda 1_{[\Lambda, +\infty)}(E_T), \quad (2)
\end{align*}
\]

with \( b(t, E_t) = b(t)E_t^\beta \) with \( \beta \in \{0, 1\} \) and \( f \) increasing.
Theoretical Existence and Uniqueness

**Theorem**

If \( \sigma(t) \geq \sigma > 0 \) then for any \( \lambda > 0 \) and \( \Lambda \in \mathbb{R} \), FBSDE (1)-(2) admits a unique solution \((E, A, Z) \in M^2\). Moreover, \( A_t \) is nondecreasing w.r.t \( \lambda \) and nonincreasing w.r.t \( \Lambda \).

**Proof**

- Approximate the singular terminal condition \( \lambda 1_{[\Lambda, +\infty)}(E_T) \) by increasing and decreasing sequences \( \{\varphi_n(E_T)\}_n \) and \( \{\psi_n(E_T)\}_n \) of smooth monotone functions of \( E_T \)
- Use
  - comparison results for BSDEs
  - the fact that \( E_T \) has a density (implying \( \mathbb{P}\{E_T = \Lambda\} = 0 \))

  to control the limits
Assume GBM for BAU emissions (Chesney-Taschini, Seifert-Uhrig-Homburg-Wagner, Grüll-Kiesel) i.e. \( b(t, e) = be \) and \( \sigma(t, e) = \sigma e \)

\[
\begin{aligned}
E_t &= E_0 + \int_0^t (b E_s - f(Y_s)) ds + \int_0^t \sigma E_s d\bar{W}_s \\
A_t &= \lambda 1_{[\Lambda, \infty)}(E_T) - \int_t^T Z_t d\bar{W}_t.
\end{aligned}
\]

Allowance price \( A_t \) constructed as \( A_t = v(t, E_t) \) for a function \( v \) which **MUST** solve

\[
\begin{aligned}
\partial_t v(t, e) + (be - f(v(t, e))) \partial_e v(t, e) + \frac{1}{2} \sigma^2 e^2 \partial_{ee} v(t, e) &= 0, \\
v(T, \cdot) &= 1_{[\Lambda, \infty)}
\end{aligned}
\]

The price at time \( t \) of a call option with maturity \( \tau \) and strike \( K \) on an allowance forward contract maturing at time \( T > \tau \) is given by

\[
V(t, E_t) = \mathbb{E}_t \{(Y_\tau - K)^+\} = \mathbb{E}_t \{(v(\tau, E_\tau) - K)^+\}.
\]

\( V \) solves:

\[
\begin{aligned}
\partial_t V(t, e) + (be - f(v(t, e))) \partial_e V(t, e) + \frac{1}{2} \sigma^2 e^2 \partial_{ee} V(t, e) &= 0, \\
V(\tau, \cdot) &= (v(\tau, \cdot) - K)^+
\end{aligned}
\]
Black-Scholes Case: \( f \equiv 0 \).

\[
\begin{align*}
 v^0(t, e) &= \lambda \mathbb{P} \left[ E_t^0 \geq \Lambda | E_t^0 = e \right] = \lambda \Phi \left( \frac{\ln (e/\Lambda e^{-b(T-t)})}{\sigma \sqrt{T-t}} - \frac{\sigma \sqrt{T-t}}{2} \right) \\
 V^0(t, e) &= \mathbb{E} \left[ (v^0(\tau, E_\tau^0) - K)^+ | E_t^0 = e \right],
\end{align*}
\]

where \( E^0 \) is the geometric Brownian motion:

\[
dE_t^0 = E_t^0 [b dt + \sigma d\tilde{W}_t].
\]

used as proxy estimation of the cumulative emissions in business as usual.
For $\epsilon \geq 0$ small, let $v^\epsilon$ and $V^\epsilon$ be the prices of the allowances and the option for $f = \epsilon f_0$. We denote by

$$v^\epsilon(T, .) = \lambda 1_{[\Lambda, \infty)}$$ and $$-\partial_t v^\epsilon - (be - \epsilon f_0(v^\epsilon))\partial_e v^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial_{ee} v^\epsilon = 0,$$

$$V^\epsilon(T, .) = (v^\epsilon(T, .) - K)^+$$ and $$-\partial_t V^\epsilon - (be - \epsilon f_0(v^\epsilon))\partial_e V^\epsilon - \frac{1}{2}\sigma^2 e^2 \partial_{ee} V^\epsilon = 0,$$

**Proposition**

As $\epsilon \to 0$, we have

$$V^\epsilon(t, s) = V^0(t, s)$$

$$+ \epsilon \mathbb{E}_{t, \epsilon} \left[ 1_{[\Lambda, \infty)}(v^0)(\tau, E^0_{\tau}) \int_t^T f_0(v^0)(s, E^0_s)\partial_e v^0(s \vee \tau, E^0_{s \vee \tau}) \frac{E^0_{s \vee \tau}}{E^0_s} ds \right]$$

$$+ o(\epsilon),$$
11 valeurs de EPSILON de 0 a 1.0

Option Prices

EUA Price

Option Price

EUA Price
A Slightly Different Model

Single good (e.g. electricity) regulated economy, with price dynamics given \textbf{exogenously}!

\[
\frac{dP_t}{P_t} = \mu(t, P_t)dt + \sigma(t, P_t)dW_t
\]

Firm $i$

- Controls its \textit{instantaneous rate of production} $q^i_t$
- \textbf{Production} over $[0, t]$

\[
Q^i_t := \int_0^t q^i_t dt.
\]

- \textbf{Costs of production} given by $c^i_t : \mathbb{R}_+ \rightarrow \mathbb{R}$ \textit{C}^1 strictly convex satisfying Inada-like conditions

\[
(c^i_t)'(0) = 0, \quad (c^i_t)'(+\infty) = +\infty
\]

- \textbf{Cumulative emissions} $E^i_t := e^i Q^i_t$
- \textbf{P&L} (wealth)

\[
X^i_t = X^i_{t,q^i,\theta^i} = x^i + \int_0^T \theta^i_t dA_t - \int_0^T [P_t q^i_t - c^i_t(q^i_t)]dt - e^i Q^i_T A_T.
\]
Proposition

If such an equilibrium exits, the optimal production strategy $\hat{q}^i$ is given by:

$$\hat{q}^i_t = [(c^i)' ]^{-1} (P_t - e^i A_t).$$

**NB:** As before the optimal production schedule $\hat{q}^i$ **DOES NOT** DEPEND upon the utility function!
Existence of Allowance Equilibrium Prices

- Set $E_t := \sum_{i \in I} E_t^i$ for the total aggregate emissions up to time $t$
- Define $f(p, y) := \sum_{i \in I} \varepsilon^i (c^i)' \varepsilon^i(p - \varepsilon^i y)$

Then the corresponding FBSDE under $\mathcal{Q}$ reads

\[
\begin{align*}
    dP_t &= \sigma(t, P_t) dB_t, \quad P_0 = p \\
    dE_t &= f(P_t, A_t) dt, \quad E_0 = 0 \\
    dA_t &= Z_t dB_t, \quad A_T = \lambda 1_{[\Lambda, +\infty)}(E_T).
\end{align*}
\]

**NB:** The volatility of the forward equation is **degenerate**!

Still, **Natural Conjecture**: For $\lambda > 0$ and $\Lambda \in \mathbb{R}$, the above FBSDE has a unique solution $(P, E, A, Z)$. 
An Enlightening Example (R.C. - Delarue)

\[
\begin{align*}
    dP_t &= dW_t, \quad P_0 = p, \\
    dE_t &= (P_t - A_t) \, dt, \quad E_0 = e \\
    dA_t &= Z_t dW_t, \quad 0 \leq t \leq T, \quad A_T = 1_{[\Lambda, \infty)}(E_T)
\end{align*}
\]  

(6)

Theorem

- There exists a unique progressively measurable triple \((P_t, E_t, A_t)_{0 \leq t \leq T}\) satisfying (6) and

\[
1_{(\Lambda, \infty)}(E_T) \leq A_T \leq 1_{[\Lambda, \infty)}(E_T).
\]

- The marginal distribution of \(E_t\)
  - is absolutely continuous for \(0 \leq t < T\)
  - has a Dirac mass at \(\Lambda\) when \(t = T\), \(\mathbb{P}\{E_T = \Lambda\} > 0\).

The terminal condition \(A_T = 1_{[\Lambda, \infty)}(E_T)\) may not be satisfied!
Comments on the Existence of a Point Mass for $E_T$

- Ruled out (by assumption) in early equilibrium model studies

**Assumption**

The $\mathcal{F}_{T-1}$-conditional distribution of $\sum_{i \in I} \Delta^i$ possesses almost surely no point mass, or equivalently, for all $\mathcal{F}_{T-1}$-measurable random variables $Z$

$$\mathbb{P}\left\{ \sum_{i \in I} \Delta^i = Z \right\} = 0 \quad (7)$$

- Thought to be *innocent*!
- We should have known better!
  - Numerical Evidence from Case Studies Shows High Emission Concentration below $\Lambda$

Carmona

Singular BSDEs
Japanese Electricity Market:

- **TOKYO** recently unveiled a **Carbon Scheme**
- Eastern & Western Regions (1GW Interconnection)
- Electricity Production: Nuclear, **Coal, Natural Gas**, Oil
  - Coal is **expensive**
  - Visible Impact of Regulation (**fuel switch**)
- **Regulation** Gory Details
  - **Cap** (Emission Target) 300 Mega-ton CO$_2$ = 20% w.r.t. 2012 BAU
  - Calibration for Fair Comparisons: **Meet Cap 95% of time**
    - Penalty 100 USD
    - Tax Level 40 USD
  - Numerical Solution of a **Stochastic Control** Problem (**HJB**) in 4-D
Yearly emissions from electricity production for the Standard Scheme, the Relative Scheme, a Tax Scheme and BAU.
Lectures based on


3. R.C., and J. Hinz: Risk-Neutral Modeling of Emission Allowance Prices and Option Valuation (working paper)