Equilibrium Pricing in Incomplete Markets

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Outline

- Mathematical finance vs. economics
- Equilibrium pricing in incomplete markets
  - The model
  - Characterization, existence and uniqueness of equilibrium
  - A generalized one fund theorem
- Computing equilibria: backward stochastic difference equations
- Investor characteristics and volatility smiles
- Explicit solutions in continuous time: forward dynamics.
- Conclusion

Mathematical Finance
vs.
Economic Theory

Some Mathematical Finance

In mathematical finance, one usually assumes that asset prices follow a diffusion (S_t) on some probability space

$$(\Omega, \mathscr{F}, \mathbb{P}).$$

In a complete market, under the assumption of no arbitrage, every claim $f(S_T)$ can be replicated and hence uniquely priced:

$$\pi_t = \mathbb{E}_{\mathbb{P}^*}[f(S_T) \mid \mathscr{F}_t] \quad \text{for a unique} \quad \mathbb{P}^* \approx \mathbb{P}.$$

Pricing uses replication and no arbitrage arguments; no reference to preferences of the market participants.

However ...

Challenge 1: Most real world markets are incomplete:

- a perfect hedge might not be possible
- risk needs to be properly accessed

Challenge 2: Non-tradable underlyings:

- weather derivatives written on temperature processes;
- CAT bonds

Challenge 3: Liquidity risk:

- trading the underlying may move the market
- sometimes liquidity is unavailable

Some Economic Theory

In general equilibrium theory one assumes that agents maximize their utility subject to their budget constraints:

$$\max_{x \in C^a} u^a(x) \quad \text{s.t.} \quad p(x) \leq p(e^a).$$

Here

- $C^a \subset C$ is the consumption set of the agent $a \in \mathbb{A}$
- e^a is her initial endowment
- $p \in C^*$ is a linear pricing functional

Prices are derived endogenously by market clearing condition

$$\sum_a (\hat{x}^a(p^*) - e^a) = 0$$

and so prices depend on preferences and endowments.

Some Economic Theory

The equilibrium approach might be more suitable for pricing weather derivatives, carbon emission credits, HOWEVER while

the case of complete markets is very well understood:

- equilibria are efficient (no waste of resources)
- existence via a representative agent approach

for incomplete markets no general theory is yet available:

- equilibria are not necessarily efficient
- representative agent approach breaks down

OFTEN VERY ABSTRACT

- EQUILIBRIA ARE DIFFICULT TO COMPUTE -

Our Approach
- The General Structure -

The Model

We consider a dynamic equilibrium model in discrete time where uncertainty is represented by a general filtered probability space

$$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0}^T, \mathbb{P}).$$

Each agent $a \in \mathbb{A}$ is endowed with an uncertain payoff

$$H^a \in L^0(\mathscr{F}_T) \quad \text{bounded from below.}$$

The agents can lend and borrow from a money market account at an exogenous interest rate. Lending and borrowing is unrestricted; all prices will be expressed in discounted terms.

Preferences

At any time $t \in \{1, 2, ..., T\}$ the agent $a \in \mathbb{A}$ maximizes a preference functional

$$U_t^a : L(\mathscr{F}_T) \to L(\mathscr{F}_t)$$

which is normalized, monotone, $\mathscr{F}_t$-translation invariant

$$U_t^a(X + Z) = U_t^a(X) + Z \quad \text{for all} \quad Z \in \mathscr{F}_t$$

$\mathscr{F}_t$-concave and (strongly) time consistent, i.e.,

$$U_{t+1}^a(X) \geq U_{t+1}^a(Y) \Rightarrow U_t^a(X) \geq U_t^a(Y).$$

Agents maximize terminal utility from trading in a financial market.

Examples of Preferences

- Expected exponential utility:

$$U_t^a(X) = -\frac{1}{\gamma_a} \ln \mathbb{E}[e^{-\gamma_a X} \mid \mathscr{F}_t].$$

- Robust expectations:

$$U_t^a(X) = \min_{P \in \mathscr{P}} \mathbb{E}_P[X \mid \mathscr{F}_t]$$

- Mean-risk preferences:

$$u_t^a(X) = \lambda \mathbb{E}[X \mid \mathscr{F}_t] - (1-\lambda)\varrho_t(X).$$

- Optimized certainty equivalents, g-expectations, ...

The Financial Market

The agents can trade a finite number of stocks and securities. The respective holdings over the time period $[t, t+1)$ are denoted

$$\eta^a_{t+1} \quad \text{and} \quad \vartheta^a_{t+1}.$$

We consider a partial (full) equilibrium model where:

- stock prices follow an exogenous stochastic process $\{S_t\}_{t=1}^T$
- securities are priced to match supply and demand
- security prices follow an endogenous process $\{R_t\}_{t=1}^T$
- securities pay a bounded dividend R at maturity hence

$$R_T = R$$

Equilibrium

An equilibrium consists of a trading strategy $\{(\hat{\eta}_t^a, \hat{\vartheta}_t^a)\}$ for every agent $a \in \mathbb{A}$ and a price process $\{R_t\}$ with $R_T = R$ s.t.:

- Individual optimality:

$$
\begin{aligned}
& U_t^a\left(H^a + \sum_{s=t}^{T-1}\{\hat{\eta}_s^a \cdot \Delta S_{s+1} + \hat{\vartheta}_s^a \cdot \Delta R_{s+1}\}\right) \\
\geq \quad & U_t^a\left(H^a + \sum_{s=t}^{T-1}\{\eta_s^a \cdot \Delta S_{s+1} + \vartheta_s^a \cdot \Delta R_{s+1}\}\right)
\end{aligned}
$$

  for all $a \in \mathbb{A}$, $t = 1, ..., T$ and $(\eta_{t+1}^a, \vartheta_{t+1}^a), ..., (\eta_T^a, \vartheta_T^a)$.

- Market clearing in the securities market:

$$
\sum_{a \in \mathbb{A}} \hat{\vartheta}_t^a = n \quad \text{for all} \quad t = 1, ..., T-1.
$$

Equilibrium

The key observation is that the evolution of utilities follows a backward dynamics. In fact, we associate with

$$\{(\eta_t^a, \vartheta_t^a)\} \quad \text{and} \quad \{R_t\}$$

the process $\{H_t^a\}$ defined by $H_T^a = H^a$ and

$$H_t^a = U_t^a \left(H^a + \sum_{s=t}^{T-1} \{\eta_s^a \cdot \Delta S_{s+1} + \vartheta_s^a \cdot \Delta R_{s+1}\} \right).$$

By translation invariance we obtain that

$$H_t^a = U_t^a \left(H_{t+1}^a + \eta_t^a \cdot \Delta S_{t+1} + \vartheta_t^a \cdot \Delta R_{t+1} \right).$$

This suggests that the dynamic problem of equilibrium pricing can be reduced to a recursive sequence of static one-period models.

Theorem (Characterization of Equilibrium)

A family of stochastic processes $\{R_t, (\hat{\eta}^a_t, \hat{\vartheta}^a_t)_{a\in\mathbb{A}}\}_{t=1}^T$ *is an equilibrium if and only if*

$$R_T = R$$

and for all $t \in \{1, 2, ..., T\}$ *the following holds:*

- *Pareto optimality:*

$$\begin{aligned} u(n) &:= \sup_{\eta^a, \sum_a \vartheta^a = n} \sum_a U^a\left(H^a_{t+1} + \eta^a_t \cdot \Delta S_{t+1} + \vartheta^a \cdot R_{t+1}\right) \\ &= \sum_a U^a_t(H^a_{t+1} + \hat{\eta}^a_t \cdot \Delta S_{t+1} + \hat{\vartheta}^a_t \cdot \Delta R_{t+1}). \end{aligned}$$

- *Prices equal marginal utility:* $R_t \in \partial u_t(n)$.
- *Security markets clear:* $\sum_a \vartheta^a_t = n$.

Theorem (Existence and Uniqueness of Equilibrium)

An equilibrium exists if the agents are sensitive to large losses, i.e.,

$$\lim_{\lambda \to \infty} U_1^a(\lambda X) = -\infty \quad \textit{if} \quad \mathbb{P}[X < 0] > 0.$$

The equilibrium is unique if the preferences are differentiable. In this case

$$S_t = \mathbb{E}_{\mathbb{Q}^a}[S_T \mid \mathscr{F}_t], \qquad R_t = \mathbb{E}_{\mathbb{Q}^a}[R_T \mid \mathscr{F}_t]$$

where

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \prod_{t=1}^{T} \nabla U_t^a \left(H^a + \sum_{t=1}^{T} \hat{\eta}_t^a \cdot \Delta S_t + \hat{\vartheta}_t^a \cdot \Delta R_t \right).$$

Theorem (One Fund Theorem)

Assume that (as in the classical CAPM)

$$U_t^a(X) = (\gamma^a)^{-1} U_t(\gamma^a X)$$

for some $\gamma^a > 0$ *and that* H^a *is "attainable", i.e.,*

$$H^a = c^a + \sum_{t=1}^{T} \tilde{\eta}_t^a \cdot \Delta S_t + \tilde{\vartheta}_t^a \cdot \Delta R_t.$$

Then in equilibrium the market participants share the gains from trading according to

$$(\gamma^a)^{-1} \left(\tilde{\eta}_t \cdot \Delta S_t + \gamma(n + \tilde{\vartheta}) \cdot \Delta R_t \right)$$

i.e., according to their risk aversion.

Theorem (One Fund Theorem Continued)

If preferences are differentiable, then the equilibrium pricing density is unique and given by

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \nabla U_t \left(\sum_{u=1}^{T} \hat{\eta}_u \cdot \Delta S_u + \gamma(n + \eta^R) \cdot \Delta R_u \right).$$

Corollary

Consider a full equilibrium model where an asset in unit net supply is priced in equilibrium. Then introducing another security in zero net supply does not alter the asset price process.

Example

Assume that all agents have exponential utility functions:

$$U_t^a(X) = -\frac{1}{\gamma_a}\mathbb{E}[e^{-\gamma_a X} \mid \mathscr{F}_t].$$

There is one stock in unit net supply that pays a dividend D at $T > 0$ and call options with different strikes on the stock in zero net supply.

By the one fund theorem the unique equilibrium pricing kernel is

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\gamma D}}{\mathbb{E}[e^{-\gamma D}]} \quad \text{where} \quad \gamma^{-1} = \sum_a (\gamma_a)^{-1}.$$

In particular, option prices for different strikes can be computed independently of each other.

Computing Equilibria
- Discrete BSDEs -

Event Trees

Let the uncertainty be generated by independent random walks

$$b_t^i = \sum_{s=1}^{t} \Delta b_s^i \quad (i = 1, ..., d).$$

Then some form of predictable representation property holds: any random variable $X \in L(\mathscr{F}_{t+1})$ can be represented as

$$X = \mathbb{E}[X|\mathscr{F}_t] + \sum_{i=1}^{M} Z^i \Delta b_{t+1}^i$$

where the random coefficients Z^i are given by

$$Z^i = \mathbb{E}[X \Delta b_{t+1}^i | \mathscr{F}_t].$$

After quite some math ...

Theorem (Equilibrium Dynamics)

There exist random functions g_t^R and g_t^a such that the equilibrium price and equilibrium processes $\{R_t\}$ and $\{H_t^a\}$ satisfy the coupled system of discrete BSDEs

$$\begin{aligned} R_t &= R_{t+1} - g^R\left((Z^a_{t+1})_{a\in\mathbb{A}}, Z^R_{t+1}\right) + Z^R_{t+1}\cdot\Delta b_{t+1} \\ H^a_t &= H^a_{t+1} - g^a(Z^a_{t+1}, Z^R_{t+1}) + Z^a_{t+1}\cdot\Delta b_{t+1} \end{aligned}$$

with terminal conditions

$$R_T = R \quad \text{and} \quad H^a_T = H^a.$$

Here Z^a_t and Z^R_t are derived from the representations of H^a_t and R_t in terms of the increments $\Delta b^1_{t+1}, \ldots, \Delta b^M_{t+1}$.

A Numerical Example

A Numerical Example

Consider a CAPM in discrete time with a stock and a put option on the stock.

The stocks pays a dividend D_T at time $T > 0$ where

$$
\begin{aligned}
\Delta D_t &= \mu h + \sqrt{v_t}\Delta b_t^1 \\
\Delta v_t &= (\kappa - \lambda v_t)h + \sigma\sqrt{v_t}\Delta b_t^2
\end{aligned}
$$

The agents have exponential utility functions and (negative) endowments in the put options:

$$
H^a = c^{a,S} S_T + c^{a,P}(K - S_T)^+
$$

The equilibrium dynamics (partial or full) can be computed; implied volatilities can be calculated.

Volatility Smiles: The Supply Effect

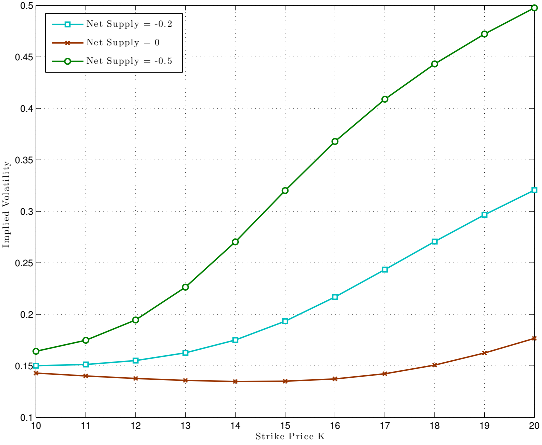

Figure: Volatility smiles in the full equilibrium model.

Volatility Smiles: The VolVol Effect

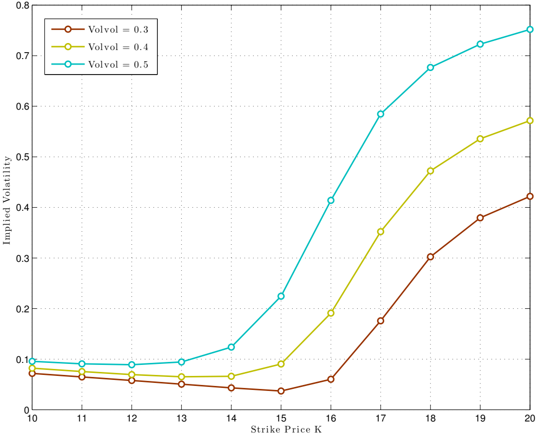

Figure: Volatility smiles in the full equilibrium model.

Stock Positions in a Partial Equilibrium Model

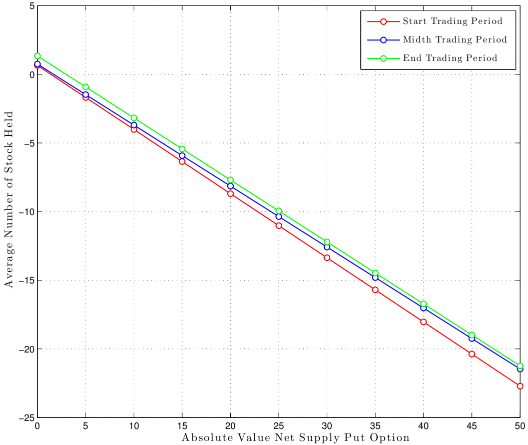

Figure: Average stock holding in a partial equilibrium model.

Outlook:
The CAPM in Continuous Time

The CAPM in Continuous Time

Consider a CAPM in continuous time with a stock in unit net supply and an option on zero net supply.

The stocks pays a dividend D_T at time $T > 0$ where

$$\begin{aligned} dD_t &= \mu dt + \sqrt{v_t} dW_t^1 \\ dv_t &= (\kappa - \lambda v_t) + \sigma \sqrt{v_t} dW_t^2 \end{aligned}$$

Due to the affine structure of $Y = (D, v)$

$$\mathbb{E}[e^{u \cdot Y} \mid \mathscr{F}_t] = e^{\phi(\tau,u) + \psi(\tau,u) \cdot Y_t}$$

for all $u \in \mathbb{C}^2$ with $\tau = T - t$. The functions (ϕ, ψ) solve a system of Riccati equations and can be given in closed form.

The CAPM in Continuous Time

Assume all agents have exponential utility functions. Then

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{e^{-\gamma S_T}}{\mathbb{E}[e^{-\gamma S_T}]}.$$

Theorem (Equilibrium Stock Prices)

The equilibrium stock price process is given by

$$S_t = \mathbb{E}_{\mathbb{Q}}[S_T \mid \mathscr{F}_t] = F(t, D_t, v_t) \mid_{u=(-\gamma,0)}$$

where

$$F(t, D, v) \quad = \quad \partial_{u_1}\phi(\tau, u) + \partial_{u_1}\psi_1(\tau, u)D + \partial_{u_1}\psi_2(\tau, u)v.$$

In particular, the equilibrium price process is given in terms of a forward dynamics.

The CAPM in Continuous Time

Equilibrium prices for options in zero net supply with different strikes are all computed under the same measure $\mathbb{Q}$:

$$C_t = \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+ \mid \mathscr{F}_t].$$

We have an integral representation for C_t that involves only the functions ϕ and ψ and the Fourier-transform of $e^{-\gamma x}(x - K)^+$.

One goal is to calibrate the local volatility

$$\frac{2\partial_\tau C_t(\tau, K)}{K^2 \partial_{KK} C_t(\tau, K)}$$

Conclusion

- Dynamic GEI model with translation invariant preferences
- Dynamic equilibrium problem can be reduced to a sequence of one-period problems
- Standard properties from the theory of complete markets carry over to incomplete markets:
    - Equilibria are (constrained) Pareto optimal
    - Equilibria exist iff a representative agent exists
    - ...
- One fund theorem with testable hypothesis
- CAPM in continuous time with explicit solution: possibility of model calibration

Thank You!