(A note) on co-integration in commodity markets

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Overview

1. Discussion of the "classical" co-integration framework
2. Co-integration in commodity spot markets, with a pricing measure $Q$
3. Implied forward prices, and their properties
4. Pricing of spread options
Co-integration in financial markets
• Co-integrated spot price model

\[ \ln S_i(t) = X(t) + Y_i(t), \ i = 1, 2 \]

\[ dX(t) = \mu \, dt + \sigma \, d\overline{B}(t) \]
\[ dY_i(t) = (c_i - \alpha_i Y_i(t)) \, dt + \eta_i \, d\overline{W}_i(t), \ i = 1, 2 \]

• \( \overline{B} \), and \( \overline{W}_i \) correlated Brownian motions
  - Short-term stationary, long-term non-stationary
  - Classical commodity spot price model (Lucia & Schwartz 2002)

• Stationary difference

\[ \ln S_1(t) - \ln S_2(t) = Y_1(t) - Y_2(t) \]
• Example: Crude oil and heating oil at NYMEX
  • Both series look non-stationary
  • and highly dependent
• The difference of (log-)prices
  • Stationary
Risk-neutral dynamics

- If spot markets are frictionless, $Q$-dynamics becomes

$$\frac{dS_i(t)}{S_i(t)} = r \, dt + \sigma \, dB(t) + \eta_i \, dW_i(t)$$

- $B$ and $W_i$ correlated Brownian motion under $Q$
  - $Q \sim P$ equivalent martingale measure
  - Girsanov’s Theorem
- No co-integration anymore!
- Spread option price: Co-integration plays no role!
  - Conclusion of Duan & Pliska 2004
• Commodity spot markets are incomplete
  - that is, trading frictions
• Extreme case: power
  - Power is non-storable
  - Similar: freight, weather
• Other cases: gas and oil
  - Storage, transportation, convenience yield
• Can co-integration be transported from $P$ to $Q$?
Co-integration in commodity spot markets
Which $Q$ should we use?

- Discussion of *risk-neutral vs. pricing* measure $Q$
  - Suppose one spot commodity, and $\overline{B}$ and $\overline{W}$ *independent* for simplicity!
  - $P$ dynamics given by,
    $$\frac{dS(t)}{S(t)} = (\tilde{\mu} - \alpha Y(t)) \ dt + \sigma \ d\overline{B}(t) + \eta \ d\overline{W}(t)$$

- Define a measure change using Girsanov ($\beta \in [0, 1]$)
  - First proposed in commodity markets by B., Cartea and Pedraz
    $$dB(t) = d\overline{B}(t) + \frac{\theta_1}{\sigma} \ dt$$
    $$dW(t) = d\overline{W}(t) - \frac{\alpha \beta Y(t)}{\eta} \ dt$$
• **Q-dynamics**

\[
\frac{dS(t)}{S(t)} = (\tilde{\mu} - \theta - \alpha(1 - \beta)Y(t)) \, dt + \sigma \, dB(t) + \eta \, dW(t)
\]

• **Special case 1**: Choose \(\beta = 1\) and \(\theta = \tilde{\mu} - r\)
  - Back to the risk-neutral case!

• **Special case 2**: Choose \(\beta = 0\)
  - Preserves the mean-reversion
  - Pricing measure \(Q\) is simply a level-shift in the mean-reversion factor

• **Note**: non-trivial to verify measure change
  - Novikov’s condition only gives validity for a fixed time horizon
  - Analogous change of measure for jumps: go to Salvador Ortiz-Latorre’s talk!
  - Produces a stochastic risk premium with changing sign
• In general, measure change will shift the level (by $\theta$), and dampen the speed of mean reversion (by $\beta \in [0, 1]$)

• Empirical evidence for this
  • B., Cartea and Pedraz: gas and oil and power

• Important observation 1
  • the mean-reversion is not killed when $0 < \beta < 1$

• Important observation 2
  • The representation of $\ln S(t)$ as a sum of a non-stationary and stationary component is preserved under $Q$
A co-integrated spot model under $Q$

- Co-integrated spot model under $Q$ based on above considerations:
  
  \[
  \ln S_i(t) = X(t) + Y_i(t), \quad i = 1, 2.
  \]

- $X$ a drifted Brownian motion
  
  \[
  dX(t) = \mu \, dt + \sigma \, dB(t)
  \]

- $Y_i$ CARMA($p$, $q$)-processes, possibly correlated with $X$
  - Generalization of the simple mean-reversion model above
  - Continuous-time autoregressive moving average process
A continuous-time ARMA($p, q$)-process

- Define the Ornstein-Uhlenbeck process $\mathbf{Z}(t) \in \mathbb{R}^p$

$$d\mathbf{Z}(t) = A\mathbf{Z}(t) \, dt + \mathbf{e}_p \sigma(t) \, dB(t),$$

- $B$ a Brownian motion (Wiener process)
- $\mathbf{e}_k$: $k$'th unit vector in $\mathbb{R}^p$, $\sigma(t)$ “volatility”
- $A$: $p \times p$-matrix

$$A = \begin{bmatrix} 0 & \cdots & 1 \\ -\alpha_p & \cdots & -\alpha_1 \end{bmatrix}$$
• Define a CAR($p$)-process as

$$Y(t) = e'_1 Z(t) = Z_1(t)$$

• More generally, a CARMA($p,q$) process for $p > q$

$$Y(t) = b' Z(t), b' = (b_0, b_1, \ldots, b_{q-1}, 1, 0, \ldots) \in \mathbb{R}^p, p > q$$

• Notice: $Y$ is stationary if and only if $A$ has eigenvalues with negative real part
• CARMA processes in weather (markets)
  • Temperature modelling: CAR(3) with seasonality (Härdle et al. 2012, B. et al. 2012))
  • Wind speed modelling: CAR(4) with seasonality (B. et al. 2012)

• CARMA processes in commodities
  • Power spot prices (EEX): CARMA(2,1) driven by a Lévy process (Garcia et al. 2010)
  • Crude oil prices: CARMA(2,1) (Paschke and Prokopczuk 2010)
Forward price dynamics
• Forward price $F_i(t, T)$ at time $t \leq T$ for a contract delivering $S_i$ at time $T$

$$F_i(t, T) = \mathbb{E}_Q[S_i(T) | \mathcal{F}_t], \ i = 1, 2$$

• Explicit price

$$F_i(t, T) = H_i(T - t) \exp \left( X(t) + b'_i e^{A_i(T-t)} Z_i(t) \right), \ i = 1, 2$$

• $H_i$ known deterministic function
  • Given by the parameters of the spot

• $F_1$ and $F_2$ not co-integrated, or are they?
We find

\[ \ln F_1(t, T) - \ln F_2(t, T) = \ln H_1(T - t) - \ln H_2(T - t) \]
\[ + b'_1 e^{A_1(T-t)}Z_1(t) - b'_2 e^{A_2(T-t)}Z_2(t) \]

Note that \( Z_i(t) \) is \( p \)-variate Gaussian distributed with constant mean and variance, asymptotically.

Using \( x = T - t \), the Musiela parametrization,

\[ \ln F_1(t, t + x) - \ln F_2(t, t + x) = \ln H_1(x) - \ln H_2(x) \]
\[ + b'_1 e^{A_1 x}Z_1(t) - b'_2 e^{A_2 x}Z_2(t) \]

Co-integrated as a process with given \textit{time-to-maturity}, but not as a process with given \textit{time-of-maturity}.
• **Forward price dynamics**

\[
\frac{dF_i(t, T)}{F_i(t, T)} = \sigma dB(t) + g_i(T - t) dW_i(t), \ i = 1, 2
\]

• Introduce the function \( g_i \)

\[
g_i(x) = \sigma_i b_i' e^{A_i x} e_p
\]

• \((F_1, F_2)\) two-dimensional geometric Brownian motion
  - Recall, \( B \), and \( W_i \) are correlated
  - Hence, \( F_1 \) and \( F_2 \) will be dependent
• Observe:
  • Co-integration in the spot is inherited as a volatility component with Samuelson effect in the two forwards
  • These two components are correlated
• Recall CARMA-processes $Y_i$ are stationary
  • $A_i$’s have eigenvalues with negative real parts
  • Hence, $g_i(x) \rightarrow 0$ as $x \rightarrow \infty$
• In the long end of the market forward prices are perfectly correlated
  \[
  \frac{dF_i(t, T)}{F_i(t, T)} \sim \sigma dB(t), \ i = 1, 2
  \]
The term structure of volatility and correlation

- Volatility term structure in \( x = T - t \), time-to-maturity
  \[
  \text{Var}(dF_i/F_i) = (\sigma^2 + 2\rho_i\sigma g_i(x) + g_i^2(x)) \, dt
  \]

  - \( \rho_i \) is the correlation between \( B \) and \( W_i \)
    - \( B \) long-term factor, \( W_i \) short term factor

- Correlation term structure
  \[
  \text{Cov}\left( \frac{dF_1}{F_1}, \frac{dF_2}{F_2} \right) = (\sigma^2 + \sigma(\rho_1 g_1(x) + \rho_2 g_2(x)) + \rho g_1(x)g_2(x)) \, dt
  \]

  - \( \rho \) is the correlation between \( W_1 \) and \( W_2 \)
    - The two short-term factors
• Numerical example: correlation structure for CAR(1) (left) and CAR(3) (right)
  • Short term factors strongly negatively correlated, long-short weakly positively correlated
  • ”Reasonable” choices of vols and mean reversions
Empirical example

- Forward prices from NYMEX
  - 3 years of daily prices for different maturities up to Feb 1, 2012
- Empirically observed correlation
Spread options on forwards: ”Margrabe-Black-76”
• Consider spread option on $F_1$ and $F_2$, with exercise time $\tau \leq T$

$$C(t, \tau, T) = e^{-r(\tau-t)}E_Q [\max(F_1(\tau, T) - F_2(\tau, T), 0) | \mathcal{F}_t]$$

• By a measure change, we can get rid of the $X$-factor in the price (Carmona and Durrleman 2003)

$$C(t, \tau, T) = e^{-r(\tau-t)}E_{\tilde{Q}} [\max(f_1(\tau, T) - f_2(\tau, T), 0) | \mathcal{F}_t]$$

$$\frac{df_i(t, T)}{f_i(t, T)} = g_i(T - t) d\tilde{W}_i, \ i = 1, 2$$

• Here, $\tilde{W}_i$ are $\tilde{Q}$-Brownian motions, correlated by $\rho$
  • Note, the spread option will not depend on $\sigma$, the long-term volatility, and its correlation with short-term variations, $\rho_i, i = 1, 2$
"Margrabe-Black-76" formula:

\[
C(t, \tau, T) = F_1(t, T)\Phi(d_1) - F_2(t, T)\Phi(d_2)
\]

where,

\[
d_1 = d_2 + \sqrt{\int_t^T g_\rho^2(T - s) \, ds},
\]

\[
d_2 = \frac{\ln F_1(t, T) - \ln F_2(t, T) - \frac{1}{2} \int_t^T g_\rho^2(T - s) \, ds}{\sqrt{\int_t^T g_\rho^2(T - s) \, ds}}
\]

and

\[
g_\rho^2(x) = g_1^2(x) - 2\rho g_1(x)g_2(x) + g_2^2(x)
\]
• For comparison: Duan-Pliska case (complete market)
• Total volatility \( g_\rho^2(x) \) substituted by

\[
\sigma^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2
\]

• Numerical example: option prices with and without co-integration
  • CAR(1)-model for the stationary part
  • Equal speed of mean reversion and short-term vol’s for both assets: \( \alpha = 0.05, \eta = 0.015 \)
  • Half life of approx. 14 days, annual vol of 24%
  • Split into strong positive and negative correlation (\( \rho = \pm 0.95 \))
  • Initial forward curves \( T \mapsto F_i(0, T) \) equal, and in either backwardation or contango: long-term level 100
• Option prices for cointegrated case as a function of $T$, time of maturity of the forwards
  • Compared with no co-integration (broken line)
  • Exercise time is $\tau = 10$
Conclusions

- Discussed co-integration in spot, and its impact on forwards and options
- Crucial feature: pricing measure preserves (parts of) the stationarity in the spots
- Forward become co-integrated in the Musiela parametrization
  - But not as processes with time of delivery given
- Analytic spread option formula: ”Margrabe-Black-76”
  - Non-stationary factor does not influence the price
- Work in progress: HJM modeling in view of these insights
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Thank you for your attention!
References

- Benth and Koekebakker (2013). A note on co-integration and spread option pricing. In progress
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