(A note) on co-integration in commodity markets

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Overview

1. Discussion of the "classical" co-integration framework
2. Co-integration in commodity spot markets, with a pricing measure Q
3. Implied forward prices, and their properties
4. Pricing of spread options

Co-integration in financial markets

- Co-integrated spot price model

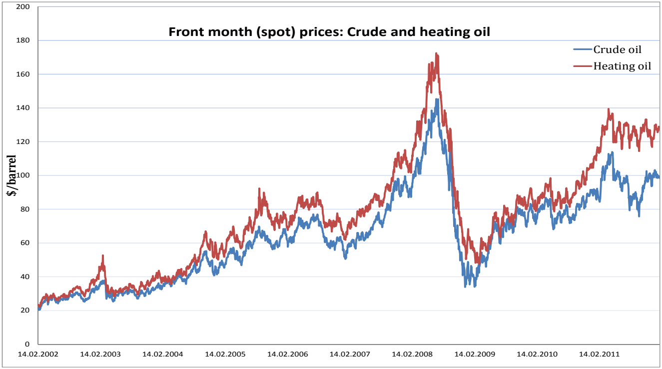
$$\ln S_i(t) = X(t) + Y_i(t)\,, i = 1, 2$$

$$dX(t) = \mu\, dt + \sigma\, d\overline{B}(t)$$
$$dY_i(t) = (c_i - \alpha_i Y_i(t))\, dt + \eta_i\, d\overline{W}_i(t)\,, i = 1, 2$$

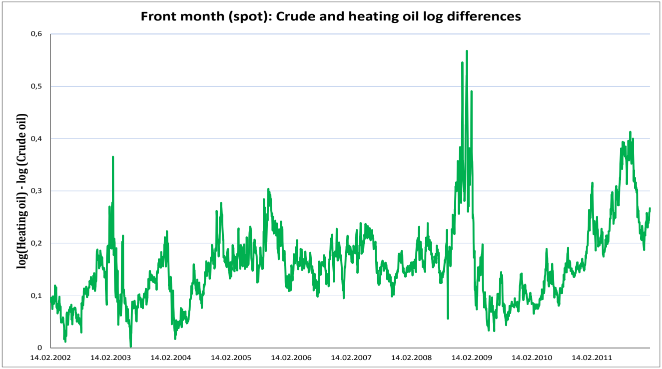
- $\overline{B}$, and $\overline{W}_i$ correlated Brownian motions
  - Short-term stationary, long-term non-stationary
  - Classical commodity spot price model (Lucia & Schwartz 2002)
- Stationary difference

$$\ln S_1(t) - \ln S_2(t) = Y_1(t) - Y_2(t)$$

- Example: Crude oil and heating oil at NYMEX
  - Both series look non-stationary
  - and highly dependent



- The difference of (log-)prices
    - Stationary



Risk-neutral dynamics

- If spot markets are *frictionless*, Q-dynamics becomes

$$\frac{dS_i(t)}{S_i(t)} = r\,dt + \sigma\,dB(t) + \eta_i\,dW_i(t)$$

- B and W_i correlated Brownian motion under Q
    - $Q \sim P$ equivalent martingale measure
    - Girsanov's Theorem
- No co-integration anymore!
- Spread option price: Co-integration plays no role!
    - Conclusion of Duan & Pliska 2004

- Commodity spot markets are incomplete
  - that is, trading frictions
- Extreme case: power
  - Power is non-storable
  - Similar: freight, weather
- Other cases: gas and oil
  - Storage, transportation, convenience yield
- Can co-integration be transported from P to Q?

Co-integration in commodity spot markets

Which Q should we use?

- Discussion of *risk-neutral* vs. *pricing* measure Q
  - Suppose one spot commodity, and $\overline{B}$ and $\overline{W}$ *independent* for simplicity!
  - P dynamics given by,

$$\frac{dS(t)}{S(t)} = (\widetilde{\mu} - \alpha Y(t))\,dt + \sigma\,d\overline{B}(t) + \eta\,d\overline{W}(t)$$

- Define a measure change using Girsanov ($\beta \in [0,1]$)
  - First proposed in commodity markets by B., Cartea and Pedraz

$$dB(t) = d\overline{B}(t) + \frac{\theta_1}{\sigma}\,dt$$

$$dW(t) = d\overline{W}(t) - \frac{\alpha\beta Y(t)}{\eta}\,dt$$

- Q-dynamics

$$\frac{dS(t)}{S(t)} = (\widetilde{\mu} - \theta - \alpha(1-\beta)Y(t))\,dt + \sigma\,dB(t) + \eta\,dW(t)$$

- Special case 1: Choose $\beta = 1$ and $\theta = \widetilde{\mu} - r$
  - Back to the risk-neutral case!
- Special case 2: Choose $\beta = 0$
  - Preserves the mean-reversion
  - Pricing measure Q is simply a level-shift in the mean-reversion factor
- Note: non-trivial to verify measure change
  - Novikov's condition only gives validity for a fixed time horizon
  - Analogous change of measure for jumps: go to Salvador Ortiz-Latorre's talk!
  - Produces a stochastic risk premium with changing sign

- In general, measure change will shift the level (by θ), and dampen the speed of mean reversion (by $\beta \in [0, 1]$)
- Empirical evidence for this
  - B., Cartea and Pedraz: gas and oil and power
- Important observation 1
  - the mean-reversion is not killed when $0 < \beta < 1$
- Important observation 2
  - The representation of $\ln S(t)$ as a sum of a non-stationary and stationary component is preserved under Q

A co-integrated spot model under Q

- Co-integrated spot model under Q based on above considerations:

$$\ln S_i(t) = X(t) + Y_i(t)\,,\, i = 1,2\,.$$

- X a drifted Brownian motion

$$dX(t) = \mu\, dt + \sigma\, dB(t)$$

- Y_i CARMA(p,q)-processes, possibly correlated with X
  - Generalization of the simple mean-reversion model above
  - Continuous-time autoregressive moving average process

A continuous-time ARMA(p, q)-process

- Define the Ornstein-Uhlenbeck process $\mathbf{Z}(t) \in \mathbb{R}^p$

$$d\mathbf{Z}(t) = A\mathbf{Z}(t)\,dt + \mathbf{e}_p\sigma(t)\,dB(t)\,,$$

- B a Brownian motion (Wiener process)
- $\mathbf{e}_k$: k'th unit vector in $\mathbb{R}^p$, $\sigma(t)$ "volatility"
- A: $p \times p$-matrix

$$A = \begin{bmatrix} \mathbf{0} & & \mathbf{I} \\ -\alpha_p & \cdots & -\alpha_1 \end{bmatrix}$$

- Define a CAR(p)-process as

$$Y(t) = \mathbf{e}_1'\mathbf{Z}(t) = Z_1(t)$$

- More generally, a CARMA(p, q) process for $p > q$

$$Y(t) = \mathbf{b}'\mathbf{Z}(t)\,, \mathbf{b}' = (b_0, b_1, \ldots, b_{q-1}, 1, 0, \ldots) \in \mathbb{R}^p, p > q$$

- Notice : Y is stationary if and only if A has eigenvalues with negative real part

- CARMA processes in weather (markets)
  - Temperature modelling: CAR(3) with seasonality (Härdle et al. 2012, B. et al. 2012))
  - Wind speed modelling: CAR(4) with seasonality (B. et al. 2012)
- CARMA processes in commodities
  - Power spot prices (EEX): CARMA(2,1) driven by a Lévy process (Garcia et al. 2010)
  - Crude oil prices: CARMA(2,1) (Paschke and Prokopczuk 2010)

Forward price dynamics

- Forward price $F_i(t,T)$ at time $t \leq T$ for a contract delivering S_i at time T

$$F_i(t,T) = \mathbb{E}_Q[S_i(T)\,|\,\mathcal{F}_t]\,, i = 1,2$$

- Explicit price

$$F_i(t,T) = H_i(T-t)\exp\left(X(t) + \mathbf{b}_i' e^{A_i(T-t)}\mathbf{Z}_i(t)\right)\,, i = 1,2$$

- H_i known deterministic function
  - Given by the parameters of the spot
- F_1 and F_2 not co-integrated, or are they?

- We find

$$\ln F_1(t,T) - \ln F_2(t,T) = \ln H_1(T-t) - \ln H_2(T-t) + \mathbf{b}_1' e^{A_1(T-t)}\mathbf{Z}_1(t) - \mathbf{b}_2' e^{A_2(T-t)}\mathbf{Z}_2(t)$$

- Note that $\mathbf{Z}_i(t)$ is p-variate Gaussian distributed with constant mean and variance, asymptotically
- Using $x = T - t$, the Musiela parametrization,

$$\ln F_1(t,t+x) - \ln F_2(t,t+x) = \ln H_1(x) - \ln H_2(x) + \mathbf{b}_1' e^{A_1 x}\mathbf{Z}_1(t) - \mathbf{b}_2' e^{A_2 x}\mathbf{Z}_2(t)$$

- Co-integrated as a process with given *time-to-maturity*, but not as a process with given *time-of-maturity*

- Forward price dynamics

$$\frac{dF_i(t,T)}{F_i(t,T)} = \sigma\, dB(t) + g_i(T-t)\, dW_i(t)\,, i=1,2$$

- Introduce the function g_i

$$g_i(x) = \sigma_i \mathbf{b}_i' e^{A_i x} \mathbf{e}_p$$

- (F_1, F_2) two-dimensional geometric Brownian motion
  - Recall, B, and W_i are correlated
  - Hence, F_1 and F_2 will be dependent

- Observe:
  - Co-integration in the spot is inherited as a volatility component with Samuelson effect in the two forwards
  - These two components are correlated
- Recall CARMA-processes Y_i are stationary
  - A_i's have eigenvalues with negative real parts
  - Hence, $g_i(x) \to 0$ as $x \to \infty$
- In the long end of the market forward prices are perfectly correlated

$$\frac{dF_i(t,T)}{F_i(t,T)} \sim \sigma\, dB(t)\,, i = 1, 2$$

The term structure of volatility and correlation

- Volatility term structure in $x = T - t$, time-to-maturity

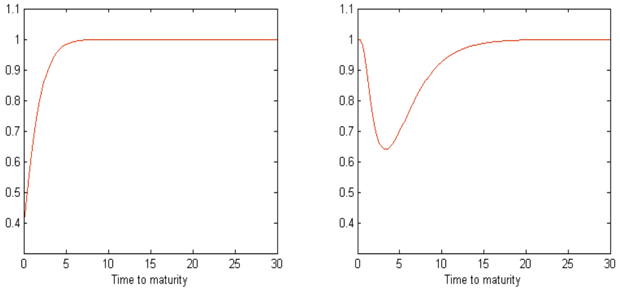
$$\mathrm{Var}(dF_i/F_i) = \boxed{(\sigma^2 + 2\rho_i \sigma g_i(x) + g_i^2(x))}\, dt$$

- ρ_i is the correlation between B and W_i
  - B long-term factor, W_i short term factor
- Correlation term structure

$$\mathrm{Cov}\left(\frac{dF_1}{F_1}, \frac{dF_2}{F_2}\right) = \boxed{(\sigma^2 + \sigma(\rho_1 g_1(x) + \rho_2 g_2(x)) + \rho g_1(x) g_2(x))}\, dt$$

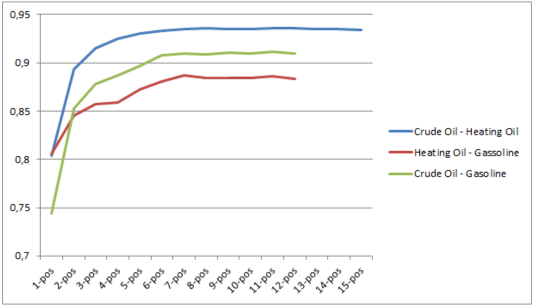
- ρ is the correlation between W_1 and W_2
  - The two short-term factors

- Numerical example: correlation structure for CAR(1) (left) and CAR(3) (right)
  - Short term factors strongly negatively correlated, long-short weakly positively correlated
  - "Reasonable" choices of vols and mean reversions



Empirical example

- Forward prices from NYMEX
  - 3 years of daily prices for different maturities up to Feb 1, 2012
- Empirically observed correlation



Spread options on forwards: "Margrabe-Black-76"

- Consider spread option on F_1 and F_2, with exercise time $\tau \leq T$

$$C(t,\tau,T) = \mathrm{e}^{-r(\tau-t)}\mathbb{E}_Q\left[\max\left(F_1(\tau,T) - F_2(\tau,T),0\right) \mid \mathcal{F}_t\right]$$

- By a measure change, we can get rid of the X-factor in the price (Carmona and Durrleman 2003)

$$C(t,\tau,T) = \mathrm{e}^{-r(\tau-t)}\mathbb{E}_{\widetilde{Q}}\left[\max(f_1(\tau,T) - f_2(\tau,T),0) \mid \mathcal{F}_t\right]$$

$$\frac{df_i(t,T)}{f_i(t,T)} = g_i(T-t)\,d\widetilde{W}_i\,,\, i = 1,2$$

- Here, $\widetilde{W}_i$ are $\widetilde{Q}$-Brownian motions, correlated by ρ
  - Note, the spread option will not depend on σ, the long-term volatility, and its correlation with short-term variations, $\rho_i, i = 1,2$

- "Margrabe-Black-76" formula:

$$C(t, \tau, T) = F_1(t, T)\Phi(d_1) - F_2(t, T)\Phi(d_2)$$

where,

$$d_1 = d_2 + \sqrt{\int_t^\tau g_\rho^2(T - s)\, ds}\,,$$

$$d_2 = \frac{\ln F_1(t, T) - \ln F_2(t, T) - \frac{1}{2}\int_t^\tau g_\rho^2(T - s)\, ds}{\sqrt{\int_t^\tau g_\rho^2(T - s)\, ds}}$$

and

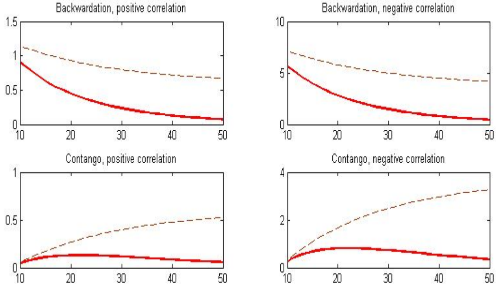
$$g_\rho^2(x) = g_1^2(x) - 2\rho g_1(x) g_2(x) + g_2^2(x)$$

- For comparison: Duan-Pliska case (complete market)
- Total volatility $g_\rho^2(x)$ substituted by

$$\sigma^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

- Numerical example: option prices with and without co-integration
  - CAR(1)-model for the stationary part
  - Equal speed of mean reversion and short-term vol's for both assets: $\alpha = 0.05, \eta = 0.015$
  - Half life of approx. 14 days, annual vol of 24%
  - Split into strong positive and negative correlation ($\rho = \pm 0.95$)
  - Initial forward curves $T \mapsto F_i(0, T)$ equal, and in either backwardation or contango: long-term level 100

- Option prices for cointegrated case as a function of T, time of maturity of the forwards
  - Compared with no co-integration (broken line)
  - Exercise time is $\tau = 10$



Conclusions

- Discussed co-integration in spot, and its impact on forwards and options
- Crucial feature: pricing measure preserves (parts of) the stationarity in the spots
- Forward become co-integrated in the Musiela parametrization
  - But not as processes with *time of delivery* given
- Analytic spread option formula: "Margrabe-Black-76"
  - Non-stationary factor does not influence the price
- Work in progress: HJM modeling in view of these insights

Thank you for your attention!

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