Continuously controlled options and related first order backward SPDEs

Nikolai Dokuchaev

Department of Mathematics & Statistics, Curtin University,

GPO Box U1987, Perth, 6845, Western Australia

Essen, October 11, 2013

We introduce a continuous time extension of swing options and Asian options such that the holder selects dynamically a continuous time process controlling the distribution of the payments (benefits) over time. Some existence results and pricing rules are obtained in Markovian and non-Markovian setting.

In Markovian setting, diffusion HJB equations are derived for modifications of Asian and swing options. In non-Markovian setting, a backward SPDE of a new type is derived for the swing options. This part is based on joint work with Christian Bender from Saarland University.

Supported by an DAAD-ATN Australia-Germany Collaboration brand and by an ARC Discovery grant.

Outline

Consider a risky asset with the price $S(t)$, where $t \in [0, T]$.

Dokuchaev (2010) introduced options with the payoff

$$F_u = \Phi(u(\cdot), S(\cdot)). \tag{1}$$

This payoff depends on a control process $u(\cdot)$ that is selected by an option holder from a certain class of admissible controls $\mathcal{U}$. The mapping $\Phi : \mathcal{U} \times \{S(\cdot)\} \to \mathbf{R}$ is given. The processes from $\mathcal{U}$ has to be adapted to the current information flow (described by the filtration $\mathcal{F}_t$).

We call the corresponding options controlled options. Clearly, an American option is a special case of controlled options.

Consider a risky asset with the price $S(t)$. Let $T > 0$ be given, and let $g : \mathbf{R} \to \mathbf{R}$ and $f : \mathbf{R} \times [0, T] \to \mathbf{R}$ be some functions. Consider an option with the payoff at time T

$$F_u = g\left(\int_0^T u(t) f(S(t), t) dt\right). \tag{2}$$

Here $u(t)$ is the control process selected by the option holder such that

$$\int_0^T u(t) dt = 1.$$

Outline

Possible selection of f and g. includes $(x-K)^+$, $(K-x)^+$, $\min(M,x)$, etc. This functions cover Asian options and limit version of swing options, where the distribution of the exercise times is continuous.

A new modification of Asian options is an option with the payoff (Dokuchaev(2010))

$$F_u = g\left(\int_0^T v(t)f(S(t),t)dt\right), \tag{3}$$

where $g(x)$ and $f(x,t)$ are given functions, the process $v(t)$ is such that

$$v(t) = \frac{u(t)}{\int_0^T u(s)ds}, \tag{4}$$

where the process $u(t)$ is adapted to the filtration $\mathcal{F}_t$ describing the flow of the available information, $u(t) \in [d_0, d_1]$, $d_0 \geq 0$.

The process $v(t)$ is not adapted, and

$$\int_0^T v(t)dt = 1.$$

In the case when $d_0 = 0$, the payoff for $u \equiv 0$ can be set by different ways. For example, it can be defined as $F_u = g\left(\frac{1}{T}\int_0^T S(t)dt\right)$, i.e., as the limit of the payoffs with $u(t) \equiv \varepsilon$ as $\varepsilon \to 0$.

Economic justification: Consider, for instance, a customer who consumes time variable and random quantity $u(t)$ of energy per time period $(t, t + dt)$, with the price $S(t)$ for a unit. The cumulated number of units consumed up to time T is $\bar{u} = \int_0^T u(t)dt$; it is unknown at times $t < T$. To hedge against the price rise, one used to purchase Asian call options with the payoff $(\bar{S} - K)^+$, where $\bar{S} = T^{-1} \int_0^T S(t)dt$. However, for tax purposes in Australia, the average price of energy for a particular customer has to be calculated as $\bar{S}_u = \bar{u}^{-1} \int_0^T u(t)S(t)dt$ rather than $\bar{S}$. Remind that $\bar{u}$ is random and unknown.

Outline

We will use the price similar to the price based on the replicability. Since the regularity of our problems is insufficient for the classical replicability, we suggest the following new definition.

Let us consider a controlled option with the payoff F_u.

Definition

The unimprovable price of an option is the price c such that

- The option writer cannot fulfill option obligations at terminal time T using the wealth raised from the initial wealth $X(0) < c$ with self-financing strategies.
- A rational option buyer would't buy an option for a higher price than c; she could rather superreplicate the payoff with some self-financing trading strategy.

Theorem

The unimprovable price c_F of an option with the payoff F_u is

$$c_F = e^{-rT} \sup_{u \in \mathcal{U}} \mathbf{E}_* F_u.$$

Outline

Consider an option with the payoff

$$F_u = g\left(\int_0^T u(t)f(S(t),t)dt\right), \tag{5}$$

where $f(x,t) : (0,+\infty) \times [0,T] \to \mathbf{R}$ and $g(x) : (0,+\infty) \to \mathbf{R}$ are given continuous non-negative functions; the function $g(x)$ is non-decreasing.

Let $\mathcal{U}$ be the class of processes $u(t)$ consisting of the processes that are adapted to the filtration $\mathcal{F}_t$ and such that

$$u(t) \in [d_0, d_1], \tag{6}$$

where $0 \leq d_0 < d_1 < +\infty$.

We consider the class $\mathcal{U}_1$ of admissible processes $u(t)$ consisting of the processes $u \in \mathcal{U}$ such that

$$\int_0^T u(t)dt = 1. \tag{7}$$

Theorem

(Dokuchaev (2010)). Assume that the function g *is concave on* $(0, +\infty)$. *In this case, an optimal control exists in* $\mathcal{U}_1$.

It follows from the definitions that the price c_F for this option can be found via solution of optimal stopping problem. For the Black-Scholes market model, this problem can be written as

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* g(x(\tau)) \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dx(t) = u(t) f(S(t), t) dt, \\ & dy(t) = u(t) dt, \\ & dS(t) = rS(t)dt + \sigma S(t) dw_*(t), \end{aligned} \tag{8}$$

where $\tau = T \wedge \inf\{t \in [0, T] : \ y(t) \geq 1\}$. In this case, $c_F = e^{-rT} \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* g(x(\tau))$ given that $x(0) = 0, \ y(0) = 0, \ S(0) = S_0$.

Alternatively, the price c_F for this option can be found via solution of optimal stochastic control problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* g(x(T)) \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dx(t) = \mathbb{I}_{\{y(t)<1\}} u(t) f(S(t),t)dt, \\ & dy(t) = u(t)dt, \\ & dS(t) = rS(t)dt + \sigma S(t)dw_*(t). \end{aligned} \tag{9}$$

In this case, $c_F = e^{-rT} \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* g(x(T))$ given that $x(0) = 0,\ y(0) = 0,\ S(0) = S_0$.

Let $f_\varepsilon(u,x,y,s)$ and g_ε be smooth approximations for $u\mathbb{I}_{\{y<1\}}f(x,t)$ and g.

Consider the stochastic control following problem:

$$\begin{aligned} &\text{Maximize} && \mathbf{E}_* g_\varepsilon(x(T)) \quad \text{over} \quad u(\cdot)\in\mathcal{U},\\ &\text{subject to} && dx(t) = f_\varepsilon(y(t),u(t),S(t),t)dt,\\ &&& dy(t) = u(t)dt,\\ &&& dS(t) = rS(t)dt + \sigma S(t)dw_*(t). \end{aligned} \tag{10}$$

Consider the corresponding value function

$$J_\varepsilon(x,y,s,t) \triangleq \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \left\{ g_\varepsilon\left(x_\varepsilon(T)\right) \Big| x(t)=x,\ y(t)=y,\ S(t)=s \right\}. \tag{11}$$

Theorem

(Dokuchaev (2010))The unimprovable option price can be found as

$$c_F = \lim_{\varepsilon \to 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{12}$$

The value function $J = J_\varepsilon$ satisfies the Bellman equation

$$\begin{aligned} &J_t + \max_{u \in [d_0, d_1]} \{J'_x f_\varepsilon + J'_y u\} + J'_s rs + \tfrac{1}{2} J''_{ss} \sigma^2 s^2 = 0, \\ &J(x, y, s, T) = g_\varepsilon(x). \end{aligned} \tag{13}$$

The Bellman equation has a unique solution.

The dimension of the Bellman equation can be reduced for the case when $g(x) \equiv x$. In this case, the option price c_F can be found via solution of optimal stopping problem

$$\begin{aligned} \text{Maximize} \quad & \mathbf{E}_* \int_0^\tau u(t) f(S(t), t) dt, \quad \text{over} \quad u(\cdot) \in \mathcal{U}, \\ \text{subject to} \quad & dy(t) = u(t) dt, \\ & dS(t) = rS(t)dt + \sigma S(t) dw_*(t), \end{aligned} \tag{14}$$

where $\tau = T \wedge \inf\{t \in [0, T] : \ y(t) \geq 1\}$. In this case, $c_F = e^{-rT} \sup_{u(\cdot)\in\mathcal{U}} \mathbf{E}_* \int_0^\tau u(t) f(S(t),) dt$ given that $y(0) = 0, \ S(0) = S_0$.

Outline

Consider an option with payoff

$$F_u = g\left(\int_0^T v(t)f(S(t),t)dt\right), \tag{15}$$

process $v(t)$ is such that

$$\int_0^T v(t)dt = 1.$$

It is formed as

$$v(t) = \frac{u(t)}{\int_0^T u(s)ds}, \tag{16}$$

Let $g_\varepsilon(x,y)$ be a smooth approximation of $g(x/y)$.

Theorem

(Dokuchaev (2010)) The nonimprovable price can be found as

$$c_F = \lim_{\varepsilon \to 0} e^{-rT} J_\varepsilon(0, 0, S(0), 0). \tag{17}$$

where J_ε *satisfies the Bellman equation*

$$\begin{aligned} &J_t + \max_{u \in [d_0, d_1]} \{J'_x f_\varepsilon + J'_y h_\varepsilon\} + J'_s rs + \tfrac{1}{2} J''_{ss} \sigma^2 s^2 = 0, \\ &J(x, y, s, T) = g_\varepsilon(x, y). \end{aligned} \tag{18}$$

The Bellman equation has a unique solution and holds a.e.

Let

$$F_u = \int_0^T u(t) f(S(t), t) dt,$$

where $f : (0, +\infty) \times [0, T] \to \mathbf{R}$ is a given function such that $|f(x, t)| \le \text{const}\,(1 + |x|)$ and $f(x, t) \ge 0$. Here $u(t)$ is the control process that is selected by the option holder. The set $\mathcal{U}_1$ of admissible processes $u(t)$ consists of the processes that are adapted to the current information flow (or to the filtration, generated by $S(t)$ and such that

$$u(t) \in [0, L], \quad \int_0^T u(t) dt \le 1.$$

This option represents the limit version of multi-exercise swing options.

Theorem

Let $f(S(t),t) = e^{r(T-t)}h(S(t))$, *where the function* $h(x)$ *is convex and non-linear in* $x > 0$, *and such that at least one of the following conditions holds:*

(i) *the function* $\alpha^{-1}h(\alpha x)$ *is non-decreasing in* $\alpha \in (0,1]$; *or*

(ii) $r = 0$.

Then $\sup_{u(\cdot)\in\mathcal{U}_1} \mathbf{E}_* F_u$ *is achieved for the control process*

$$\widehat{u}(t) = \begin{cases} L, & t \geq T - 1/L \\ 0, & t < T - 1/L, \end{cases}$$

The price of the option is

$$e^{-rT}\mathbf{E}_*\int_0^T \widehat{u}(t)f(S(t),t)dt = \frac{e^{-rT}}{L}\mathbf{E}_*\int_{T-1/L}^T f(S(t),t)dt.$$

Outline

Let $X(t)$ be a current payoff process: a random RCLL process (right continuous with left limits) process, $X(t) \geq 0$ a.s. and that $\mathbf{E} \sup_{t \in [0,T]} X(t)^2 < +\infty$. For example, we may consider $X(t) = (K - S(t))^+$, where $S(t)$ is the energy price.

Let $\mathcal{F}_t$ be the filtration generated by $X(t)$.

Let $U(t)$ be the set of processes $u(s) : [t, T] \times \Omega \to \mathbf{R}$ adapted to $\mathcal{F}_t$ and such that $u(s) \in [0, L]$. Let $U(t, y)$ be the set of processes $u(\cdot) \in U(t)$ such that $\int_t^T u(s)ds \leq 1 - y$ a.s..

Consider an option with payoff

$$F(u, t) = \int_t^T u(s)X(s)ds, \tag{19}$$

Let y be a $\mathcal{F}_t$-measurable random variable with values in $[0,1]$, and let

$$J(t,y) = \operatorname*{ess\,sup}_{u \in U(t,y)} \mathbf{E}_t F(u,t), \tag{20}$$

where $\mathbf{E}_t = \mathbf{E}\{\cdot|\mathcal{F}_t\}$ and $t \in [0,T]$.

Theorem

For any $(t,y) \in [0,T] \times [0,1]$, there exists optimal $u^{t,y}(\cdot) \in U(t,y)$.

Theorem

One sided derivatives $D_y^{\pm}J(y,t)$ are defined and coincides a.e.

Theorem

(Dokuchaev and Bender (2013)). BSDE for J:

$$J(y,t) = L\mathbf{E}_t \int_t^T (D_y^{\pm}J(s,y) + X(s))_+ ds,$$
$$J(s,1) = 0.$$

In particular,

$$J(T,y) = 0$$

Theorem

For $y \in [0,1]$, $t < T - (y-1)/L$, *the control process* $u(s) = u^{y,t}(s)$ *and the corresponding process* $y(s) = y + \int_t^s u(r)dr$ *are optimal if and only if*

$$\begin{aligned} u(s) = 0 \quad & if \quad D_y^- J(s, y(s), \omega) + X(s,\omega) < 0 \\ u(s) = L \quad & if \quad D_y^- J(s, y(s), \omega) + X(s,\omega) > 0, \end{aligned} \tag{21}$$

up to equivalency in (t, ω).

- Dokuchaev, N. (2010). Controlled Options: Derivatives with Added Flexibility. Working paper: http://ssrn.com/abstract=1720957 (also, available on ArXiv). Published in (2013) at Int. J. Theor. Appl. Finance 16.
- Benth, F. E., Lempa, J., Nilssen, T. K. (2012): On the optimal exercise of swing options in electricity markets. Journal of Energy Markets 4, 3–28. (presented in EF Conference 2010).
- Basei, M., Cesaroni, A., Vargiolu, T. (2013): Optimal exercis of swing contracts in energy markets: an integral constrained stochastic optimal control problem. Preprint available on arXiv.
- Bender, C.,, Dokuchaev, N. (2013): A first-order BSPDE for swing option pricing. http://arxiv.org/abs/1305.3988.