On Forward Modelling In Electricity Markets: An Infinite Dimensional Stochastic Analysis Perspective

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I am gratefull for support from the MAWREM project.

Contents

1. Modelling futures in electricity markets
2. Spot and futures dynamics when the noise term is Gaussian or normal inverse Gaussian
3. Representing the futures by a sum of OU-type processes

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- Derivatives: Futures $F(t, T_1, T_2)$ with delivery period $[T_1, T_2]$, $t \in [0, T_2]$ with $F(T_2, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(r) dr$. Futures are more traded than the underlying itself.

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- Instataneous future rates in Musiela parametrisation

$$f(t,x) := E(S(t+x)|\mathcal{F}_t) = \lim_{h \searrow 0} F(t, t+x, t+x+h), \quad t,x \in \mathbb{R}_+$$

where the expectation is to be taken under the pricing measure.

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We use the Hilbert space

$$H_\alpha := \{f : \mathbb{R}_+ \to \mathbb{C} : f' e_{\alpha/2} \in L^2([0,\infty))\}$$

endowed with the scalar product
$\langle f, g \rangle_\alpha := f(0)g(0) + \int_0^\infty f'(y)g'(y)e_\alpha(y)dy$ where $e_\alpha(x) = \exp(\alpha x)$ and $\alpha > 0$.
We consider the general dynamics under the pricing measure

$$df(t) = \partial_x f(t)dt + \Psi(t)dL(t)$$

where L is some square integrable mean zero Lévy process and $\Psi \in \mathcal{L}_L^2$.

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- H_α is a Banach algebra relative to the pointwise multiplication, i.e. the pointwise multiplication is bilinear and continuous.
- Hilbert-Schmidt operators on H_α can be classified completely. They are a sum of a one-dimensional operator and an integral operator.
- The space-derivative ∂_x is the generator of the shifting semigroup $(U_t)_{t\geq 0}$, i.e. $U_t g(x) = g(x+t)$, $t, x \geq 0$, $g \in H_\alpha$. $(U_t)_{t\geq 0}$ is a quasi contractive strongly continuous semigroup.

Mild solutions to SPDEs

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$$\Gamma : \mathbb{R}_+ \times H_w \to L(H_w)$$

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and $f_0 \in H_\alpha$. Then there is a càdlàg process f with values in H_α such that

$$f(t) = U_t f_0 + \int_0^t U_{t-s}\Gamma(s, f(s))dL(s)$$

where $(U_t)_{t\geq 0}$ is the shifting semigroup. For each $g \in \partial_x^*$ we also have

$$\langle g, f(t)\rangle = \langle g, f_0\rangle + \int_0^t \langle \partial_x^* g, f(s)\rangle ds + \int_0^t \langle g, \Gamma(s, f(s))dL(s)\rangle \quad t \geq 0.$$

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Note: The Lipschitz conditions on Γ can be weakend substantially! Cf. [Tappe, 12]

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for some Lévy processes L_n and some $g_n \in H_\alpha$. Moreover, $F(t, T_1, T_2) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \mathcal{E}(N_y)(t)dy$ where N_x is a time-inhomogenous Lévy process and $\mathcal{E}(N_x)$ denotes its stochastic exponential.

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Theorem

Let $f(t) = U_t f_0 + \int_0^t U_{t-s}\Psi(s)dW(s)$ *where* $f_0 \in H_\alpha$, W *is a (possibly with values in a separable infinite dimensional Hilbert space) Brownian motion with covariance* Q *and* $\Psi \in \mathcal{L}^2_W(H_\alpha)$ *a suitable operater valued integrand.*

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$$S(t) = f_0(t) + \int_0^t k(t,s)dB(s)$$

where B *is a standard Brownian motion,*

$$k^2(t,s) := (\Psi(s)Q\Psi^*(s)h_{t-s})(t-s)$$

and $h_t \in H_w$ *is given by* $h_y(x) = \frac{1}{\alpha}(1 - e^{-\alpha(x \wedge y)}) + 1$.

Hilbert space valued normal inverse Gaussian process (HNIG)

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Definition

Let U be a Hilbert space and L be a U-valued Lévy process. L is an HNIG process if $\langle u, L(1) \rangle$ is normal inverse Gaussian distributed for any $u \in U$.

Remark: HNIG processes can be characterised completely.

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$$S(t) = f_0(t) + \int_0^t k(t,s)dN(s)$$

where N *is a normal inverse Gaussian process,*

$$k^2(t,s) := \beta(\Psi(s)Q\Psi^*(s)h_{t-s})(t-s)$$

for some $\beta \geq 0$ *and* $h_t \in H_w$ *is given by* $h_y(x) = \frac{1}{\alpha}(1 - e^{-\alpha(x \wedge y)}) + 1$.

Dynamics of 2 Futures

Corollary

Let $f(t) = U_t f_0 + \int_0^t U_{t-s}\Psi(s)dW(s)$ *where* $f_0 \in H_\alpha$, W *is a (possibly with values in an separable infinite dimensional Hilbert space) Brownian motion with covariance* Q *and* $\Psi \in \mathcal{L}^2_W(H_\alpha)$ *a suitable operater valued integrand. Let* $0 \leq t \leq T_1 \leq T_2$ *and* $X(t) := (F(t, T_1), F(t, T_2))$*. Then we have*

$$X(t) = (f_0(T_1), f_0(T_2)) + \int_0^t m(s)dB(s)$$

for some 2*-dimensional standard Brownian motion* B *and*

$$(m(s)^2)_{ij} = \langle y_i, U_{T_1-s}\Psi(s)Q\Psi(s)^* U^*_{T_1-s} y_j\rangle, \quad i, j = 1, 2$$

for some known functions y_1, y_2.

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A nice subspace of H_α

Lemma

Let $x_0 > 0$. Then there is a closed subspace $H_\alpha^{x_0}$ of H_α such that the following statements hold.

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1. *There is a continuous projection* $\Pi_{x_0} : H_\alpha \to H_\alpha^{x_0}$ *such that* $\Pi_{x_0} g(x) = g(x)$ *for any* $g \in H_\alpha$*,* $x \in [0, x_0]$*.*

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2. *$H_\alpha^{x_0}$ is invariant under the shift semigroup $(U_t)_{t\geq 0}$.*
3. *$H_\alpha^{x_0}$ has a Riesz basis $(g_n)_{n\geq 0}$ such that $g_0(x) = 1$, $x \in \mathbb{R}_+$ and $g_n(x) = \frac{1}{\lambda_n \sqrt{x_0}}(1 - e^{-\lambda_n x})$, $x \in \mathbb{R}_+, n \geq 1$ for a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{C}$.*

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4. *If $(g_n)_{n\geq 0}$ is as in (3), $(g_n^*)_{n\geq 0}$ the corresponding biorthogonal system, then $U_t^* g_n^* = e^{-\lambda_n t} g_n^*$, $n \geq 1$, $t \geq 0$ and $g_0^* = g_0$.*

OU-type representation

Theorem

Assume that the futures price process $(f(t))_{t\geq 0}$ *has values in* $H_w^{x_0}$ *for some* $x_0 > 0$.

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$$f(t) = g_0 S(t) + \sum_{n=1}^{\infty} g_n \int_0^t e^{(s-t)\lambda_n} dM_n(s), \quad t \in \mathbb{R}_+.$$

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where the sum converges almost surely in H_α *and* $(g_n)_{n\geq 0}$ *is the Riesz basis provided in the theorem before.*
The complex valued martingales are given by

$$M_n(t) = \int_0^t \langle g_n^*, \Psi(s) dL(s)\rangle, \quad t \geq 0.$$

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Thank you for your attention!