A pricing measure to explain risk premium in power markets

Salvador Ortiz-Latorre

joint work with Fred S. Benth

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Stylized facts of the electricity spot prices

- Seasonal behaviour in yearly, weekly and daily cycles.
- Approximate stationary behaviour: Mean reversion.
- Non-Gaussianity and extreme spikes.
- Historical spot price at NordPool from the beginning in 1992 (NOK/MWh).
Factor models for the spot price

- We can consider two kind of models:
  - The arithmetic spot price model, defined by
    \[ S(t) = \Lambda_a(t) + X(t) + Y(t), \quad t \in [0, T^*]. \]
  - The geometric spot price model, defined by
    \[ S(t) = \Lambda_g(t) \exp(X(t) + Y(t)), \quad t \in [0, T^*]. \]
  - \( \Lambda_a(t) \) and \( \Lambda_g(t) \) are assumed to be deterministic and they account for seasonalities in the prices.
  - \( X(t) \) has continuous paths and explains *normal variations*.
  - \( Y(t) \) has jumps and accounts for the *spikes*.
  - \( X(t) \) and \( Y(t) \) are mean reverting stochastic processes.
The (instantaneous) forward price

- In practice, electricity is a non-storable commodity.
- There is no buy and hold strategies $\implies$ classical non-arbitrage arguments break down.
- **Incomplete market**: any probability measure $Q$ equivalent to the historical measure $P$ is valid.
- The forward price with time to delivery $0 < T < T^*$ at time $0 < t < T$ is given by

$$F_Q(t, T) = \mathbb{E}_Q[S(T) | \mathcal{F}_t]$$

where $\mathcal{F}_t$ is the information in the market up to time $t$.
- The (theoretical) risk premium for forward prices is defined by the following expression

$$R_Q^F(t, T) \triangleq \mathbb{E}_Q[S(T) | \mathcal{F}_t] - \mathbb{E}_P[S(T) | \mathcal{F}_t].$$
Risk premium profile

- If $R^F_Q(t, T) > 0$, the market is in "contango".
- If $R^F_Q(t, T) < 0$, the market is in "normal backwardation".
- **Goal**: To reproduce the following risk premium profile.
Mathematical modeling of the factors

- Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\) be a complete filtered probability space, where \(T > 0\) is a fixed finite time horizon.

- Consider a standard Brownian motion \(W\) and a pure jump Lévy subordinator

\[
L(t) = \int_0^t \int_0^\infty zN^L(ds, dz), \quad t \in [0, T],
\]

where \(N^L(ds, dz)\) is a Poisson random measure with Lévy measure \(\ell\) satisfying \(\int_0^\infty z\ell(dz) < \infty\).

- Let \(\kappa_L(\theta) \triangleq \log \mathbb{E}_{\mathbb{P}}[e^{\theta L(1)}]\) and

\[
\Theta_L \triangleq \sup\{\theta \in \mathbb{R}_+ : \mathbb{E}[e^{\theta L(1)}] < \infty\}.
\]

- A minimal assumption is that \(\Theta_L > 0\).

- In the geometric spot model we also need \(\Theta_L > 1\).
Mathematical modeling of the factors

Consider the Ornstein-Uhlenbeck processes

\[ X(t) = X(0) - \alpha_X \int_0^t X(s) \, ds + \sigma_X \, W(t), \]
\[ Y(t) = Y(0) + \int_0^t (\kappa'_L(0) - \alpha_Y Y(s)) \, ds + \int_0^t \int_0^\infty z \tilde{N}^L(ds, dz), \]

with \( t \in [0, T], \alpha_X, \sigma_X, \alpha_Y > 0, X(0) \in \mathbb{R}, Y(0) \geq 0. \)

Using Itô formula one gets the following explicit representation

\[ X(T) = X(t) e^{-\alpha_X(T-t)} + \sigma_X \int_t^T e^{-\alpha_X(T-s)} \, dW(s), \]
\[ Y(T) = Y(t) e^{-\alpha_Y(T-t)} + \frac{\kappa'_L(0)}{\alpha_Y} \left( 1 - e^{-\alpha_Y(T-t)} \right) + \int_t^T \int_0^\infty e^{-\alpha_Y(T-s)} \, z \tilde{N}^L(ds, dz), \]

where \( 0 \leq t \leq T. \)
The change of measure

- For \( t \in [0, T] \), consider the following family of Wiener and Poisson integrals

\[
\tilde{G}_{\theta_1, \beta_1}(t) \triangleq \int_0^t \sigma_X^{-1}(\theta_1 + \alpha_X \beta_1 X(s)) \, dW(s),
\]

\[
\tilde{H}_{\theta_2, \beta_2}(t) \triangleq \int_0^t \int_0^\infty \left( e^{\theta_2 z} \left( 1 + \frac{\alpha_Y \beta_2}{\kappa''(\theta_2)} z Y(s-) \right) - 1 \right) \tilde{N}^L(ds, dz),
\]

where \( \bar{\beta} \in [0, 1]^2, \bar{\theta} \in \bar{D}_L \triangleq \mathbb{R} \times D_L \) and \( D_L \triangleq (-\infty, \Theta_L/2) \).

- The desired family of parametrised measure changes \( Q_{\bar{\theta}, \bar{\beta}} \) is given by the following Radon-Nykodim density

\[
\frac{dQ_{\bar{\theta}, \bar{\beta}}}{dP}_{\mathcal{F}_t} \triangleq \mathcal{E}(\tilde{G}_{\theta_1, \beta_1} + \tilde{H}_{\theta_2, \beta_2})(t) = \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(t) \mathcal{E}(\tilde{H}_{\theta_2, \beta_2})(t),
\]

\( t \in [0, T] \), where \( \mathcal{E}(\cdot) \) denotes the stochastic exponential.
The change of measure

- Recall that, if $M$ is a semimartingale, the stochastic exponential of $M$ is given by

\[
\mathcal{E}(M)(t) = \exp \left( M(t) - \frac{1}{2}[M^c, M^c](t) \right) \\
\times \exp \left( - \sum_{0 \leq s \leq t} \Delta M(s) - \log(1 + \Delta M(s)) \right).
\]

- If $M$ is a local martingale, then $\mathcal{E}(M)$ is also a local martingale.
- If $\mathcal{E}(M)$ a positive local martingale, then $\mathcal{E}(M)$ is a supermartingale and $\mathbb{E}_P[\mathcal{E}(M)(t)] \leq 1$, $t \in [0, T]$.
- To have a well defined change of measure we need to ensure that $\mathbb{E}_P[\mathcal{E}(M)(T)] = 1$ and $\mathcal{E}(M)(t) > 0$, $t \in [0, T]$.
- Classical sufficient criteria do not provide sharp conditions.
The change of measure

Sketch of the proof that $\mathcal{E}(M)$ is a martingale with $M = \tilde{G}_{\theta_1, \beta_1}$ or $\tilde{H}_{\theta_2, \beta_2}$:

- Localise $\mathcal{E}(M)(t)$ using a reducing sequence $\{\tau_n\}_{n \geq 1}$.
- For any $n \geq 1$, $\{\mathcal{E}(M)(t)^{\tau_n}\}_{t \in [0, T]}$ is a true martingale and induces a change of measure.
- Test the uniform integrability of $\{\mathcal{E}(M)(T)^{\tau_n}\}_{n \geq 1}$ with $G(x) = x \log(x)$, i.e.

$$\sup_n \mathbb{E}_P \left[ G(\mathcal{E}(M)(T)^{\tau_n}) \right] < \infty.$$ 

- But this can be rewritten as

$$\sup_n \mathbb{E}_{Q^n} \left[ \log(\mathcal{E}(M)(T)^{\tau_n}) \right] < \infty.$$ 

- We can get rid off the ordinary exponential in $\mathcal{E}(M)(T)^{\tau_n}$.
- The problem is reduced to finding a uniform bound for the second moment of $X$ and $Y$ under $Q^n$. 

The dynamics under the new pricing measure

- Using a general version of Girsanov’s theorem we get the dynamics for $X$ and $Y$.
- The $Q_{\tilde{\theta},\tilde{\beta}}$-compensator measure of $Y$ is given by

$$
\nu_{Q_{\tilde{\theta},\tilde{\beta}}}^Y(dt, dz) = e^{\theta_2 z} \left( 1 + \frac{\alpha_Y \beta_2}{\kappa''_L(\theta_2)} z Y(t-) \right) \ell(dz) dt.
$$

- Using Itô’s formula again we get

$$
X(T) = X(t) e^{-\alpha_X(1-\beta_1)(T-t)} + \frac{\theta_1}{\alpha_X(1-\beta_1)} (1 - e^{-\alpha_X(1-\beta_1)(T-t)}) \\
+ \sigma_X \int_t^T e^{-\alpha_X(1-\beta_1)(T-s)} dW_{Q_{\tilde{\theta},\tilde{\beta}}}(s),
$$

$$
Y(T) = Y(t) e^{-\alpha_Y(1-\beta_2)(T-t)} + \frac{\kappa'_L(\theta_2)}{\alpha_Y(1-\beta_2)} (1 - e^{-\alpha_Y(1-\beta_2)(T-t)}) \\
+ \int_t^T \int_0^\infty e^{-\alpha_Y(1-\beta_2)(T-s)} z \tilde{N}_{Q_{\tilde{\theta},\tilde{\beta}}}^L(ds, dz),
$$

where $0 \leq t \leq T$. 
Forward price formula for arithmetic spot model

- Recall that $S(t) = \Lambda_a(t) + X(t) + Y(t), \quad t \in [0, T^*].$

**Theorem**

*The forward price $F_Q(t, T)$ in the arithmetic spot model is given by*

$$F_{Q_{\theta, \beta}}(t, T) = \Lambda_a(T) + X(t)e^{-\alpha_X(1-\beta_1)(T-t)} + Y(t)e^{-\alpha_Y(1-\beta_2)(T-t)}$$

$$+ \frac{\theta_1}{\alpha_X(1-\beta_1)}(1 - e^{-\alpha_X(1-\beta_1)(T-t)})$$

$$+ \frac{\kappa'_L(\theta_2)}{\alpha_Y(1-\beta_2)}(1 - e^{-\alpha_Y(1-\beta_2)(T-t)}).$$

- This pricing formula allows to model the spot price with stationary factors and obtain non-deterministic forward prices on the long end of the forward curve.
Risk premium formula for arithmetic spot model

**Theorem**

The risk premium for the forward price in the arithmetic spot model is given by

\[ R_{a,Q_{\theta,\hat{\beta}}}(t, T) = X(t) e^{-\alpha_X(T-t)}(e^{\alpha_X \beta_1(T-t)} - 1) \]
\[ + Y(t) e^{-\alpha_Y(T-t)}(e^{\alpha_Y \beta_2(T-t)} - 1) \]
\[ + \frac{\theta_1}{\alpha_X(1-\beta_1)}(1 - e^{-\alpha_X(1-\beta_1)(T-t)}) \]
\[ + \frac{\kappa_L'\left(\theta_2\right)}{\alpha_Y(1-\beta_2)}(1 - e^{-\alpha_Y(1-\beta_2)(T-t)}) - \frac{\kappa_L'(0)}{\alpha_Y}(1 - e^{-\alpha_Y(T-t)}). \]

- Analysis of possible risk profiles in **Benth and O.-L. (2013)**.
Geometric spot model

- The problem is reduced to compute $\mathbb{E}_{\bar{\theta}, \bar{\beta}}[\exp(Y(T))|\mathcal{F}_t]$.
- Fortunately, $Y(t)$ is an affine process!!!.
- We have that

$$\mathbb{E}_{\bar{\theta}, \bar{\beta}}[\exp(Y(T))|\mathcal{F}_t] = \exp \left( Y(t) \Psi_{\theta_2, \beta_2}(T - t) + \Psi_{\theta_2, \beta_2}(T - t) \right),$$

$t \in [0, T]$, where the pair of functions $\Psi_{\theta_2, \beta_2}^1$ and $\Psi_{\theta_2, \beta_2}^0$ is the solution of the so called generalised Riccati equation. See Kallsen and Muhle-Karbe (2010).
- Nonlinear ODE depending on the Lévy measure and the parameters.
- We classify the possible behaviour of the solutions in terms of the parameters, as we are only interested in global solutions.
- The expression for the forward price is obtained and a theoretical analysis of the possible risk profiles is performed in Benth and O.-L. (2013).
References


