Optimal exercise of swing contracts in energy markets: an integral constrained stochastic optimal control problem

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Outline

1. The problem
2. Swing contracts with penalties
3. Swing contracts with strict constraints
4. Conclusions

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Swing contracts in continuous time

The paper of Benth-Lempa-Nilssen (2012) was a real breakthrough in the evaluation of swing contracts in continuous time (also *Best Paper Award* of E&F 2010 Essen).

It characterized the value of a swing contract with final constraint $Z(T) \in [m, M]$, with $m = 0$, as the value function of a stochastic control problem, linked to a suitable Hamilton-Jacobi-Bellman equation $\rightsquigarrow$ nice numerics. . .

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1. from the mathematical side: HJB equations seldom (and not in this case) have explicit solutions to be checked to be smooth: if not, in which sense the value function can be a solution? Is it in our case?
2. from the applied side: "true" contracts have $m > 0$ (often $m > 0.5M$). Is it still possible, for this situation, to write down a HJB equation? With which boundary conditions?

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More in details...

Following Benth-Lempa-Nilssen (2012), for a given contract with maturity T the buyer can choose, at each time $s \in [0, T]$, a marginal amount of energy $u(s) \in [0, \bar{u}]$ at a prespecified strike price K.

- Marginal profit & loss: $(P(s) - K)u(s)$, where $P(s)$ is the spot price of "energy".
- Cumulative P&L:

$$\int_0^T e^{-rs}(P(s) - K)u(s)\, ds,$$

with $r > 0$ risk-free interest rate.

However, the seller usually wants the total amount of energy $Z(T) = \int_0^T u(s)\, ds$ to lie between a minimum and a maximum quantity, i.e. $Z(T) \in [m, M]$.

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Two kinds of constraints

The constraint $Z(T) \in [m, M]$ is implemented in two main ways.

1. **penalties:** make the buyer pay a penalty $\widetilde{\Phi}(P(T), Z(T))$, where $\widetilde{\Phi}(p, z)$ is null for $z \in [m, M]$ and usually convex in z.
2. **strict constraint:** impose the constraint $Z(T) \in [m, M]$ to be satisfied strictly, i.e. to force the buyer to withdraw the minimum cumulative amount of energy m and to stop giving the energy when the maximum M has been reached (BLN 2012, but only with $m = 0$).

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Stochastic control of swings with penalties

In the case of swing contracts with penalties, we get a standard stochastic control problem, as the maximization of the final expected payoff for a buyer entering in the contract at a generic time $t \in [0, T]$ is given by

$$\widetilde{V}(t,p,z) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} (P(s) - K) u(s)\, ds - e^{-r(T-t)} \widetilde{\Phi}(P(T), Z(T)) \right],$$

with $(t,p,z) \in [0,T] \times \mathbb{R}^2$, where $\mathbb{E}_{tpz}$ stands for the expectation conditioned to $P(t) = p$, $Z(t) = z$ and

$$\mathcal{A}_t := \{ u = \{u(s)\}_{s \in [t,T]} \mid u \text{ progr. meas. and s.t. } u(s) \in [0, \bar{u}] \}$$

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Stochastic control of swings with strict constraints

Conversely, swing contracts with strict constraints give rise to a stochastic control problem with a nonstandard state constraint:

$$V(t,p,z) = \sup_{u \in \mathcal{A}^{adm}_{tz}} \mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P(s)-K)u(s)\, ds \right], \qquad (1)$$

with (t,p,z) in a suitable domain $\mathcal{D} \subseteq [0,T] \times \mathbb{R}^2$, where

$$\mathcal{A}^{adm}_{tz} := \{u \in \mathcal{A}_t \mid Z(T) \in [m,M] \quad \mathbb{P}_{tpz}\text{-a.s.}\}$$

Due to the presence of the constraint on $Z^{t,z;u}(T)$, here standard theory does **not** apply.

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The link between the two kind of contracts

Swing contracts with penalties or with strict constraints are not so different:

- intuitively, if you fix a high penalty, the buyer will have little interest in violating the constraint $Z(T) \in [m, M]$;
- mathematically, we prove that in a set $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$ the value function V pricing a swing with strict constraint is the limit of the value functions V^c pricing suitable unconstrained contracts, where the constraint has been substituted by an appropriate penalization in the pricing functional.
- moreover, both the value functions are solutions of a Hamilton-Jacobi-Bellman (HJB) equation with suitable boundary conditions.

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Swing contracts with penalties

Let $T > 0$ be a fixed contract horizon and $t \in [0, T]$. We model the price of energy through a stochastic process P which satisfies the SDE

$$dP^{t,p}(s) = f(s, P^{t,p}(s))ds + \sigma(s, P^{t,p}(s))dW(s), \quad s \in [t, T], \tag{2}$$

with initial condition $P^{t,p}(t) = p$, where $f, \sigma \in C([0, T] \times \mathbb{R}; \mathbb{R})$ are suitable functions. Also, for each $s \in [t, T]$ and $u \in \mathcal{A}_t$, denote by $Z^{t,z;u}(s)$ the energy bought up to time s:

$$Z^{t,z;u}(s) = z + \int_t^s u(\tau)d\tau, \qquad s \in [t, T].$$

If the globally purchased energy $Z^{t,z;u}(T)$ does not fall within a fixed range $[m, M]$ ($m, M \in \mathbb{R}$, with $m \leq M$), the holder must pay a penalty $\widetilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))$, with $\widetilde{\Phi} : \mathbb{R}^2 \to \mathbb{R}$.

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Various kinds of penalty

Typical examples:

1. penalty directly proportional to $P^{t,p}(T)$ and to the entity of the overrunning or underrunning:

$$\widetilde{\Phi}(p,z) = -Ap(z-M)^{+} - Bp(m-z)^{+},$$

   for all $(p,z) \in \mathbb{R}^2$, where $A, B > 0$ are suitable constants. In several practical cases, $A = B$.
2. replace p above, representing the spot price at the end T of the contract, with an arithmetic mean of spot prices over $[0,T]$ (thus requiring another state variable in the problem).
3. replace p with a fixed (high) penalty.

In the light of the above discussion, we assume that $\widetilde{\Phi}$ is null for $z \in [m,M]$, globally concave in z and such that

$$|\widetilde{\Phi}(p+h,z)-\widetilde{\Phi}(p,z)| \le Ch(1+|z|), \quad |\widetilde{\Phi}(p,z+h)-\widetilde{\Phi}(p,z)| \le Ch(1+|p|),$$

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The stochastic control problem

Let $r \geq 0$ be the risk-free rate. We get a stochastic optimal control problem, with the following value function:

$$\widetilde{V}(t,p,z) = \sup_{u \in \mathcal{A}_t} \widetilde{J}(t,p,z;u), \tag{3}$$

for each $(t,p,z) \in [0,T] \times \mathbb{R}^2$, where

$$\begin{aligned}\widetilde{J}(t,p,z;u) \quad = \quad & \mathbb{E}_{tpz}\Big[\int_t^T e^{-r(s-t)}(P^{t,p}(s)-K)u(s)ds + \\ & + e^{-r(T-t)}\widetilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))\Big]\end{aligned}$$

Problem (3) belongs to a widely studied class of control problems, with well-known classical results.

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The HJB equation

Theorem. The function $\widetilde{V}$ is the unique viscosity solution of

$$-\widetilde{V}_t + r\widetilde{V} - f\widetilde{V}_p - \frac{1}{2}\sigma^2\widetilde{V}_{pp} + \min_{v\in[0,\bar{u}]}[-v(\widetilde{V}_z + p - K)] = 0, \qquad \forall (t,p,z) \in [0,T[\times\mathbb{R}^2, \quad (4)$$

with final condition

$$\widetilde{V}(T,p,z) = \widetilde{\Phi}(p,z), \qquad \forall (p,z) \in \mathbb{R}^2, \quad (5)$$

and such that

$$|\widetilde{V}(t,p,z)| \leq \check{C}(1+|p|^2+|z|^2), \qquad \forall (t,p,z) \in [0,T]\times\mathbb{R}^2,$$

for some constant $\check{C} > 0$.

Remark. Equation (4) is the same as in BLN 2012, but with different domain and boundary conditions (5).

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Properties of the value function

Proposition. For each $(t, z) \in [0, T] \times \mathbb{R}$ the function $\widetilde{V}(t, \cdot, z)$ is Lipschitz continuous, uniformly in t. Moreover, the derivative $\widetilde{V}_p(t, p, z)$ exists for a.e. $(t, p, z) \in [0, T] \times \mathbb{R}^2$ and we have $|\widetilde{V}_p(t, p, z)| \leq M_1(1 + |z|)$, for some constant $M_1 > 0$.

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Monotonicity of the value function

The last monotonicity result is described in Figure 1.

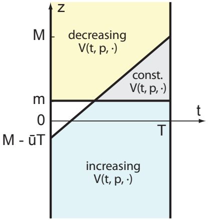

Figure: monotonicity of $\widetilde{V}(t, p, \cdot)$

Apparently unexpected result: for suitable t and for all p, the function $V(t, p, \cdot)$ is constant in an interval.

As a matter of fact, if z is in the grey region, then $Z^{t,z;u}(T) \in [m, M]$ for each $u \in \mathcal{A}_t$, so that $\widetilde{\Phi}(P(T), Z(T)) = 0$ and the initial z does not influence the value function.

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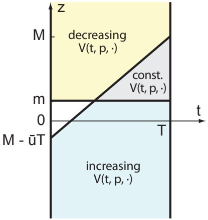

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Optimal buying strategy

As already observed in BLN 2012, an optimal control policy is

$$\underline{u}(t,p,z) = \begin{cases} \bar{u} & \text{if } \widetilde{V}_z(t,p,z) \geq p - K, \\ 0 & \text{if } \widetilde{V}_z(t,p,z) < p - K. \end{cases} \quad (6)$$

Notice that, by our regularity results, the candidate in (6) is a.e. well-defined.

Moreover, since $\widetilde{V}$ is concave in z, for each fixed (t,p) there exists $\bar{z}(t,p) \in [-\infty, +\infty]$ such that $\widetilde{V}_z(t,p,z) < p - K$ if and only if $z > \bar{z}(t,p)$: for t fixed, the function $\bar{z}(t,\cdot)$ (which in BLN 2012 is called **exercise curve**) can be used to write $\underline{u}$ as

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Stochastic control problem

Let P and Z be as before. This time we get a stochastic optimal control problem with the following value function:

$$\widetilde{V}(t,p,z) = \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \widetilde{J}(t,p,z;u), \tag{8}$$

where

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$$\mathcal{A}_{tz}^{\text{adm}} := \{u \in \mathcal{A}_t \mid Z(T) \in [m,M] \quad \mathbb{P}_{tpz}\text{-a.s.}\}$$

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Admissible domain

In order for $\mathcal{A}^{adm}_{tz}$ to be nonempty, we must impose that (t, p, z) belongs to

$$\mathcal{D} = \{(t, p, z) \in [0, T] \times \mathbb{R}^2 : m - \bar{u}(T - t) \leq z \leq M\}$$

We also will need the following sets (the latter for $\rho > 0$):

$$\widetilde{\mathcal{D}} = \{(t, p, z) \in [0, T] \times \mathbb{R}^2 : m - \bar{u}(T - t) < z < M\},$$

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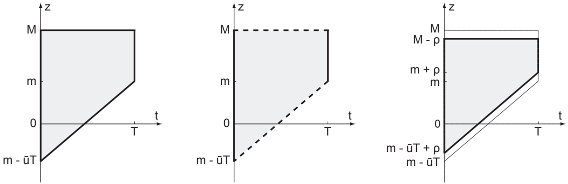

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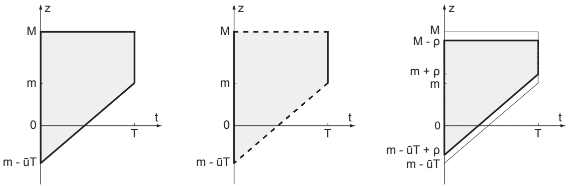

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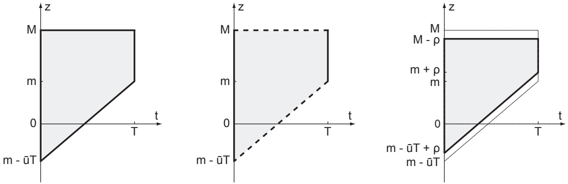

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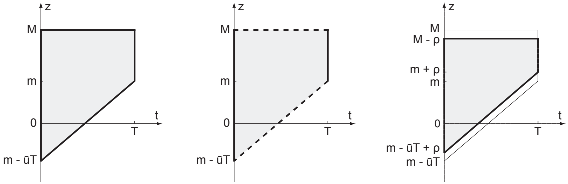

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The idea

Approximate the value function $\widetilde{V}$, pricing a swing with strict constraint, as the limit of value functions $\widetilde{V}^c$ pricing unconstrained contracts with the penalization

$$\Phi^c(p,z) = -c\left[\left(z-\left(M-\frac{1}{\sqrt{c}}\right)\right)^+ + \left(\left(m+\frac{1}{\sqrt{c}}\right)-z\right)^+\right],$$

Nice economical interpretation: here we are approximating a swing contract with the strict constraint $Z(T) \in [m, M]$ with a sequence of suitable contracts with increasing penalties for $Z(T) \notin \left[m+\frac{1}{\sqrt{c}}, M-\frac{1}{\sqrt{c}}\right]$.

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The approximating problems

Let

$$\begin{aligned}\alpha &= \{(t,p,z) \in \mathcal{D} : z = M\},\\ \beta &= \{(t,p,z) \in \mathcal{D} : z + \bar{u}(T-t) = m\},\\ \gamma &= \{T\} \times \mathbb{R} \times [m, M],\end{aligned}$$

so that $\mathcal{D} \setminus \widetilde{\mathcal{D}} = \alpha \cup \beta$.

Proposition. The functions $\widetilde{V}^c$ converge to $\widetilde{V}$ uniformly on compact subsets of $\widetilde{\mathcal{D}}$, and $\widetilde{V}$ is continuous in $\mathcal{D}$. Moreover, if $(t,p,z) \in \alpha$ we have $\widetilde{V}(t,p,z) = 0$, and if $(t,p,z) \in \beta$ we have

$$\widetilde{V}(t,p,z) = \bar{u}\,\mathbb{E}_{tpz}\left[\int_t^T e^{-r(s-t)}(P^{t,p}(s) - K)ds\right] =: \xi(t,p)$$

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the question is: will $\widetilde{V}$ solve the same equation?

Theorem. The function V is the unique viscosity solution of Equation (9) in the domain $\mathcal{D} \setminus (\alpha \cup \beta \cup \gamma)$, with boundary conditions

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$$|V(t,p,z)| \leq \check{C}(1 + |p|^2 + |z|^2), \qquad \forall (t,p,z) \in \mathcal{D}, \qquad (10)$$

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Proposition. For each $(t, p) \in [0, T] \times \mathbb{R}$ the function $V(t, p, \cdot)$ is:

- concave, Lipschitz continuous and a.e. twice differentiable;
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Properties of the value function

Proposition. Let $(t, z) \in [0, T] \times \mathbb{R}$ be such that $(t, p, z) \in \mathcal{D}$ for each $p \in \mathbb{R}$. Then:

- the function $V(t, \cdot, z)$ is Lipschitz continuous, uniformly in (t, z). Moreover, the derivative $V_p(t, p, z)$ exists for a.e. $(t, p, z) \in \mathcal{D}$ and we have $|V_p(t, p, z)| \leq M_1$, for some constant $M_1 > 0$.
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Monotonicity of the value function

The last monotonicity result is described in Figure 3.

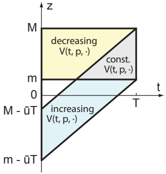

Figure: monotonicity of $V(t, p, \cdot)$

As in the previous case, it was foreseeable that the function $V(t, p, \cdot)$ is constant in an interval: if $M - (T - t)\bar{u} \geq m$ and $z \in [m, M - (T - t)\bar{u}]$ then $\mathcal{A}^{adm}_{tz} = \mathcal{A}_t$ (i.e. all controls satisfies the constraint), which implies that the initial value z does not influence the value function. This generalizes an intuitive result in BLN 2012: for (t, z) such that the volume constraint is *de facto* absent, the value function V does not depend on z.

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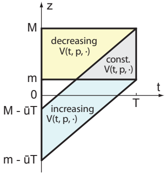

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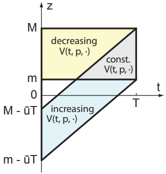

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Optimal buying strategy

An optimal control policy is again

$$\underline{u}(t,p,z) = \begin{cases} \bar{u} & \text{if } \widetilde{V}_z(t,p,z) \geq p - K, \\ 0 & \text{if } \widetilde{V}_z(t,p,z) < p - K. \end{cases} \tag{11}$$

Notice that also in this case, by our regularity results, $\underline{u}$ is a.e. well-defined. Since $\widetilde{V}$ is again concave in z, for each fixed (t,p) there exists $\bar{z}(t,p) \in [-\infty, +\infty]$ such that $\widetilde{V}_z(t,p,z) < p - K$ if and only if $z > \bar{z}(t,p)$: for t fixed, the **exercise curve** $\bar{z}(t,\cdot)$ can be used to write $\underline{u}$ as

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- We characterize the value of swing contracts in continuous time as the unique viscosity solution of a HJB equation with suitable boundary conditions.
- The case of contracts with penalties is a straightforward application of classical optimal control theory: their value is the unique viscosity solution of a HJB equation, and in that case only a terminal condition is needed.
- Conversely, the case of contracts with strict constraints is a nonstandard stochastic control problem. We approximate the value function with a sequence of value functions of appropriate penalized swing contracts and show that they converge to the value of a contract with strict constraints. The value function is also the unique viscosity solution of the same HJB equation as before, subject to terminal and boundary conditions.

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