Asian basket options and implied correlations in oil markets

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- **Basket option**: option whose underlying is a *basket* (i.e. a portfolio) of assets.
- Particular case of a basket option: *spread* option

- Payoff of a European basket call option: $\left(B(T)-X\right)^{+}$

  $B(T)$ is the basket value at the time of maturity T,

  X is the strike price.

- Payoff of an Asian basket call option:

  B(T) is replaced by *A(T):* the average basket value between times 0 and *T*.

Commodity baskets and spreads

- Crack spread:

$$k_u * \textit{Unleaded gasoline} + k_h * \textit{Heating oil} - \textit{Crude}$$

- Energy company portfolios:

$$k1 * E1 + k2 * E2 + \ldots + kn\ En$$

where *ki*'s can be positive as well as negative (i.e. a portfolio can contain both *long* and *short* positions).

Motivation:

- Commodity portfolios contain two or more assets, and often contain both *long and short positions.*
- The valuation and hedging of options on such portfolios (i.e. *basket options*) is challenging because

  the sum of lognormal r.v.'s is not lognormal.
- Such portfolios can have negative values, so ***lognormal distribution cannot be used, even in approximation***.
- Most existing approaches can only deal with regular basket options or options on a spread between two assets (Kemna and Vorst, Kirk, Turnbull and Wakeman, ...).
- Numerical and Monte Carlo methods are slow, do not provide closed formulae.
- Need to extend pricing and hedging to Asian basket options.

GLN (Generalized Lognormal) approach:

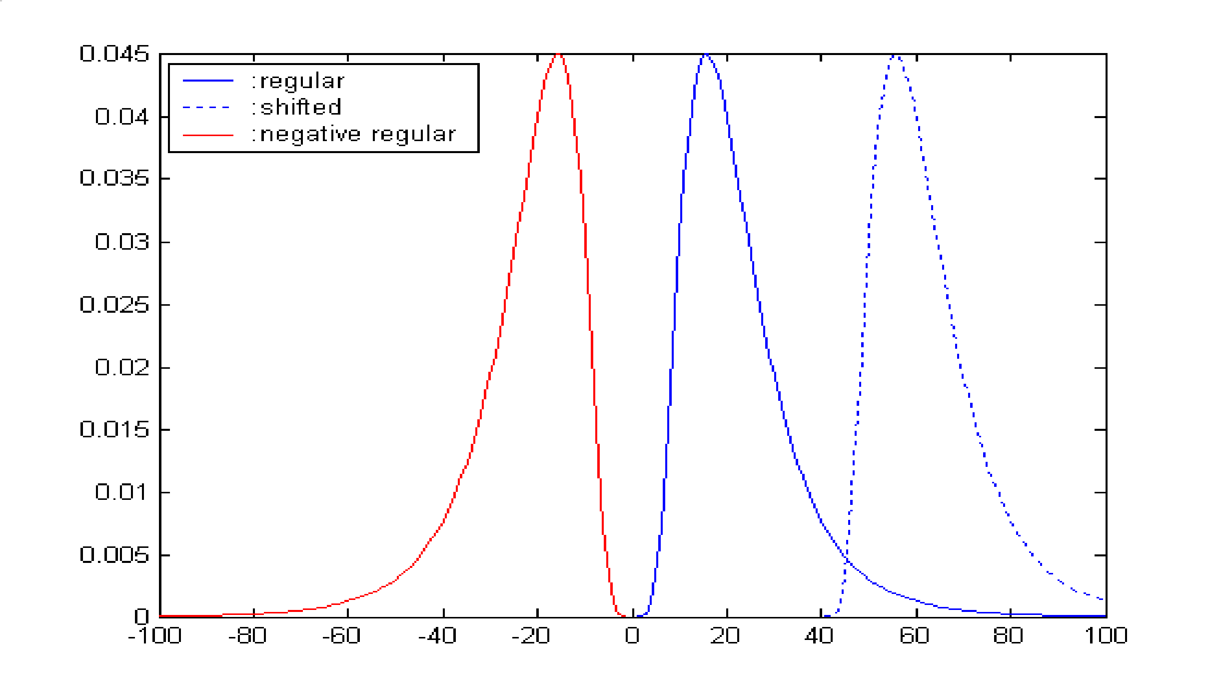
- Essentially a *moment-matching method*.
- Portfolio (i.e. basket) distribution is approximated using a *generalized family of lognormal distributions* :

shifted or negative shifted lognormal distribution

The main attractions:

- applicable to options on portfolio with several long and short positions
- naturally extendable to Asian-style options
- allows to apply Black-Scholes formula
- provides closed formulae for the option price and the greeks

Regular lognormal, shifted lognormal and negative regular lognormal



Assumptions:

- Portfolio consists of futures on different (but related) commodities. The portfolio's value at time of option maturity *T*

$$B(T) = \sum_{i=1}^{N} a_i . F_i(T)$$

where a_i : the weight of asset (futures contract) *i*,

N : the number of assets in the portfolio,

$F_i(T)$: the futures price *i* at the time of maturity .

- The futures in the portfolio and the option on it mature on the same date.

Individual assets' dynamics:

Under the risk adjusted probability measure Q, the futures prices are martingales. The stochastic differential equations for $F_i(t)$ is

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i . dW^{(i)}(t),\; i = 1,2,3,\ldots,N$$

where

$F_i(t)$:the futures price *i* at time *t*

σ_i :the volatility of asset *i*

$W^{(i)}(t), W^{(j)}(t)$:the Brownian motions driving assets *i* and *j* with correlation $\rho_{i,j}$

The first three moments and the skewness of the basket on maturity date T can be calculated:

$$E\ B(T) = M_1(T) = \sum_{i=1}^{N} a_i.F_i(0)$$

$$E\ (B(T))^2 = M_2(T) = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i.a_j.F_i(0).F_j(0).\exp\left(\rho_{i,j}.\sigma_i.\sigma_j.T\right)$$

$$E\ (B(T))^3 = M_3(T) =$$

$$= \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i.a_j.a_k.F_i(0).F_j(0).F_k(0).\exp\left(\rho_{i.j}.\sigma_i.\sigma_j.T + \rho_{i.k}.\sigma_i.\sigma_k.T + \rho_{j.k}.\sigma_j.\sigma_k.T\right)$$

$$\eta_{B(T)} = \frac{E\left(B(T) - E(B(T))\right)^3}{\sigma_{B(T)}^3}$$

where $\sigma_{B(T)}$: standard deviation of basket at the time T

- If we assume the distribution of a basket is shifted lognormal with parameters m, s, τ , the parameters should satisfy non-linear equation system :

$$M_1(T) = \exp\left((m + \frac{1}{2}s^2 \right)$$

$$M_2(T) = \tau^2 + 2.\tau.\exp\left(m + \frac{1}{2}s^2 \right) + \exp(2m + 2s^2)$$

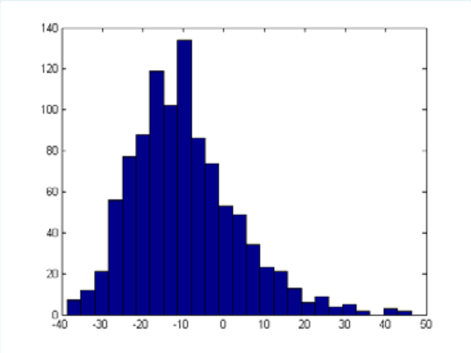
$$M_3(T) = \tau^3 + 3.\tau^2.\exp\left(m + \frac{1}{2}s^2 \right) + 3.\tau.\exp(2m + 2s^2) + \exp\left(3m + \frac{9}{2}s^2 \right)$$

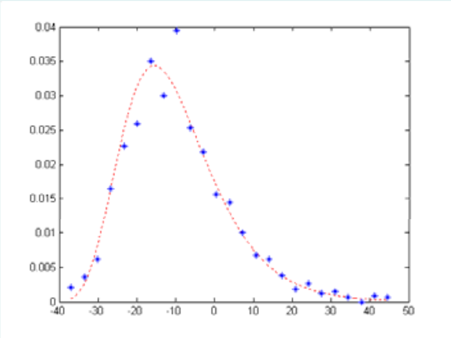
- If we assume the distribution of a basket is negative shifted lognormal, the parameters should satisfy non-linear equation system above by changing $M_1(T)$ to $-M_1(T)$ and $M_3(T)$ to $-M_3(T)$.

Approximating distribution:

Skewness		$\eta > 0$		$\eta < 0$
Approximating distribution		***shifted***		***negative shifted***

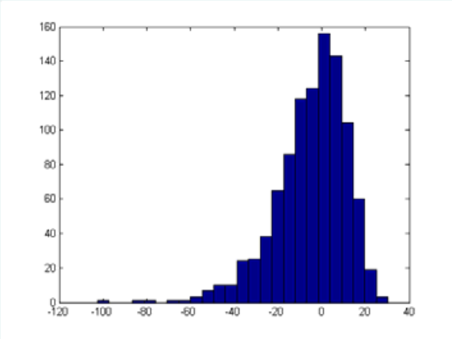
Examples of terminal basket value distribution:

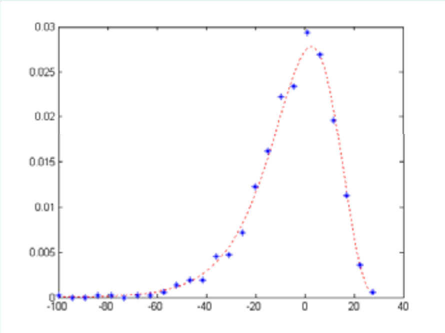




Shifted lognormal

$$Fo = [100;90];\ \sigma = [0.2;0.3];\ a = [-1;1];\ X = -10;\ r = 3\%;\ T = 1\ year;\ \rho = 0.9$$





Negative shifted lognormal

$$Fo = [105;100];\ \sigma = [0.3;0.2];\ a = [-1;1];\ X = -5;\ r = 3\%;\ T = 1\ year;\ \rho = 0.9$$

Valuation of a European call option (shifted lognormal):

- Suppose that the distribution of basket 1 is lognormal.
  Then an option on such a basket can be valued by applying the Black-Scholes (actually Black's (1975)) formula.

- Suppose that the relationship between basket 2 and basket 1 is

$$B^{(2)}(t) = B^{(1)}(t) + \tau$$

- The payoff of a call option on basket 2 with the strike price X is:

$$\left(B^{(2)}(T) - X\right)^{+} = \left(\left(B^{(1)}(T) + \tau\right) - X\right) = \left(B^{(1)}(T) - (X - \tau)\right)^{+}$$

It is the payoff of a call option on basket 1 with the strike price $(X - \tau)$

Valuation of call option (negative lognormal):

- Suppose again that the distribution of basket 1 is lognormal (an option on such a basket can be valued by applying the Black-Scholes formula).

- Suppose that the relationship between basket 2 and basket 1 is

$$B^{(2)}(t) = -B^{(1)}(t)$$

- The payoff of a call option on basket 2 with the strike price X is:

$$\left(B^{(2)}(T) - X\right)^{+} = \left(-B^{(1)}(T) - X\right) = \left(\left(-X\right) - B^{(1)}(T)\right)^{+}$$

It is the payoff of a put option on basket 1 with the strike price $-X$

Closed form formulae for a (European) basket call option:

- For e.g. shifted lognormal :

$$c = \exp(-rT)\left[(M_1(T)-\tau).N(d_1)-(X-\tau).N(d_2)\right]$$

where $$d_1 = \frac{\log(M_1(T)-\tau)-\log(X-\tau)+\frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(M_1(T)-\tau)-\log(X-\tau)-\frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T)-2.\tau.M_1(T)+\tau^2}{(M_1(T)-\tau)^2}\right)}$$

It is the call option price with strike price $(X-\tau)$.

Differentiate it w.r.t. parameters → analytic expressions for the greeks

Asian baskets

Underlying: average basket value over a certain interval

Note:

$$A_B(T) = \frac{1}{n}\sum_{k=t_1}^{t_n} B(t_k) = \frac{1}{n}\sum_{k=t_1}^{t_n}\sum_{i=1}^{N} a_i F_i(t_k) = \sum_{i=1}^{N} a_i A_i(T)$$

So the average basket value is simply the *basket of individual assets' averages*, with the same weights

→ assets' averages are approximated by lognormal distributions, by matching first two moments (as in Wakeman method)

→ the GLN approach then applies directly, only with different moments (calculated from the moments of the average asset prices)

→ closed-form expressions for option prices and greeks.

	Basket 1	***Basket 2***	***Basket 3***	***Basket 4***	***Basket 5***	***Basket 6***
Futures price (Fo)	**[100;120]**	**[150;100]**	**[110;90]**	**[200;60]**	**[95;90;105]**	**[100;90;95]**
Volatility (σ)	**[0.2;0.3]**	**[0.3;0.2]**	**[0.3;0.2]**	**[0.3;0.2]**	**[0.2;0.3;0.25]**	**[0.25;0.3;0.2]**
Weights (a)	**[-1;1]**	**[-1;1]**	**[0.7;0.3]**	**[-1;1]**	**[1; -0.8; -0.5]**	**[0.6;0.8; -1]**
Correlation (ρ)	**0.9**	**0.3**	**0.9**	**0.9**	$\rho_{1,2} = \rho_{2,3} =$ **0.9** $\rho_{1,3} =$ **0.8**	$\rho_{1,2} = \rho_{2,3} =$ **0.9** $\rho_{1,3} =$ **0.8**
Strike price (X)	**20**	**-50**	**104**	**-140**	**-30**	**35**
skewness (η)	$\eta > 0$	$\eta < 0$	$\eta > 0$	$\eta < 0$	$\eta < 0$	$\eta > 0$
Location parameter (τ)	$\tau < 0$	$\tau < 0$	$\tau > 0$	$\tau > 0$	$\tau < 0$	$\tau < 0$

T=1 year; r = 3 %

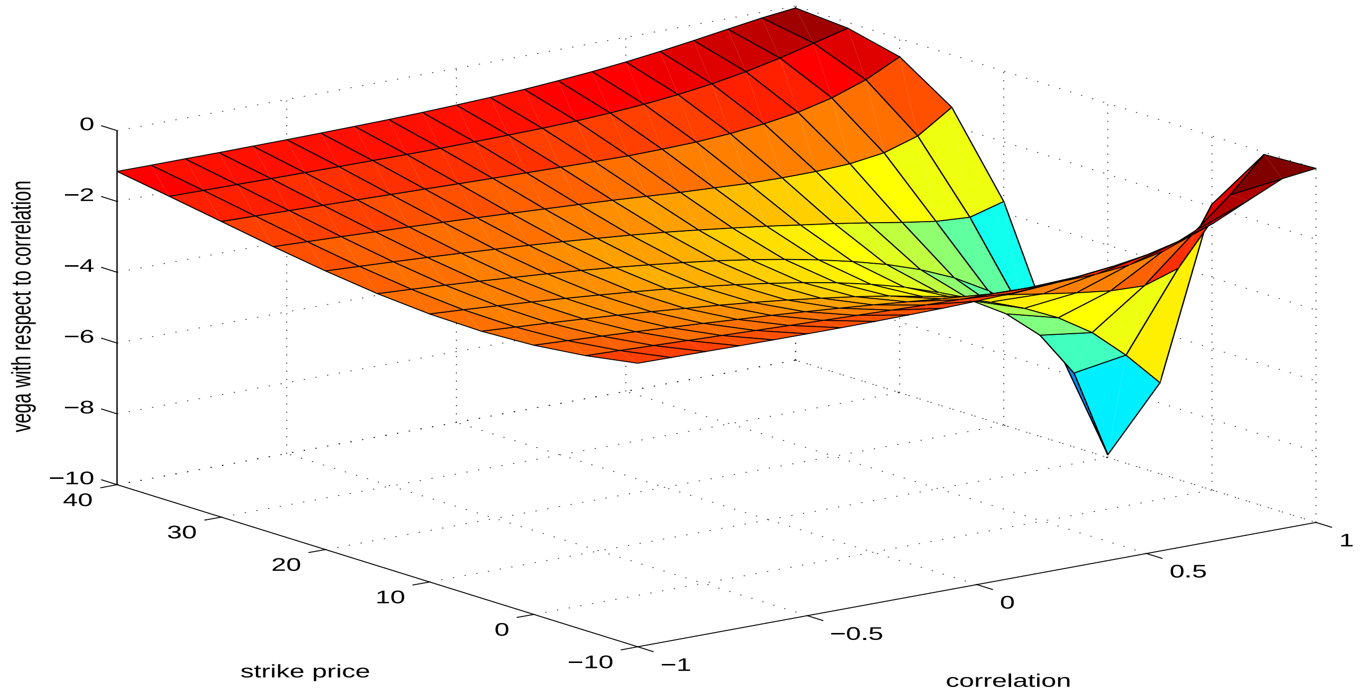
Simulation results (Asian call option prices)

Method	*Basket 1*	*Basket 2*	*Basket3*	*Basket 4*	*Basket 5*	*Basket 6*
GLN	**5.9** *neg. shifted*	**12.8** *neg. shifted*	**8.25** *shifted*	**14.5** *neg. shifted*	**5.95** *neg. shifted*	**7.1** *neg. shifted*

Monte Carlo	5.9 *(0.11)*	12.75 *(0.02)*	8.22 *(0.01)*	14.5 *(0.02)*	5.95 *(0.02)*	7.06 (0.03)

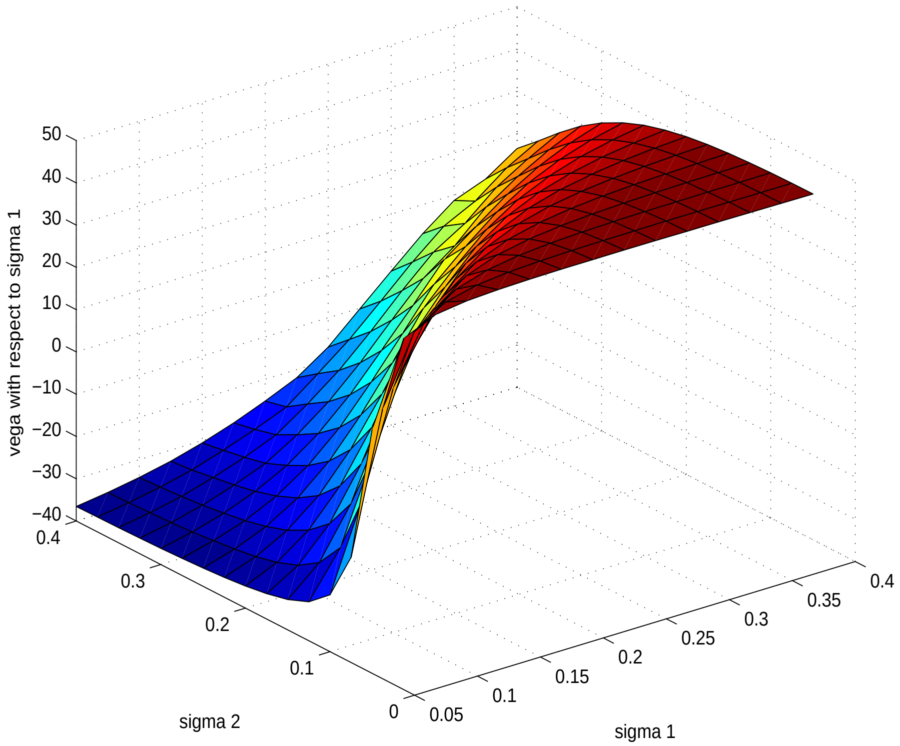
Greeks: correlation vega

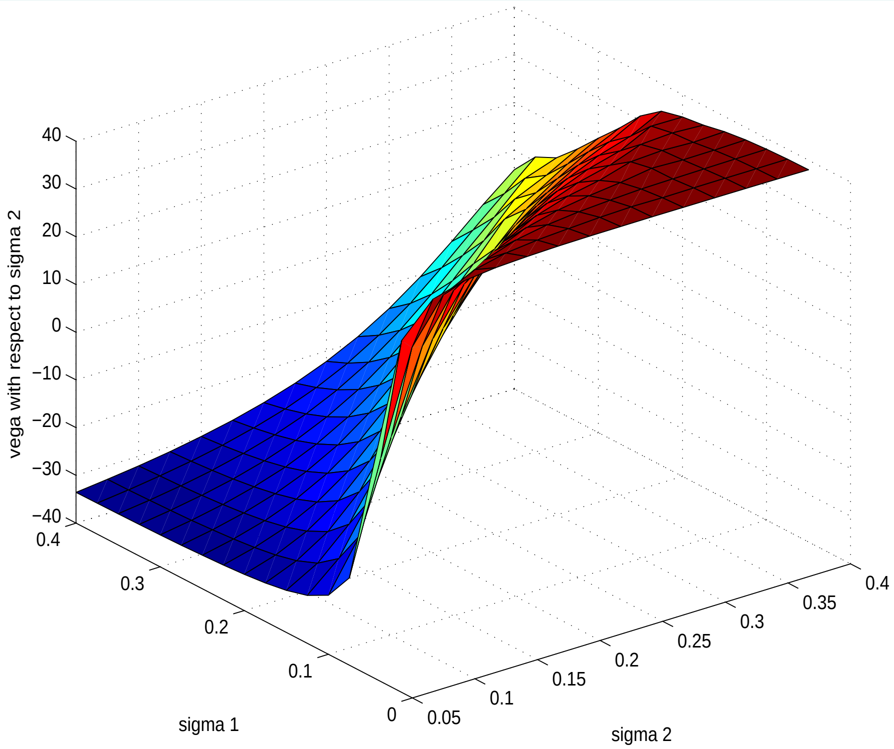
Spread [100,120], vols=[0.2,0.3]



Volatility vegas vs. volatilities

same spread, X=20, correlation=0.9





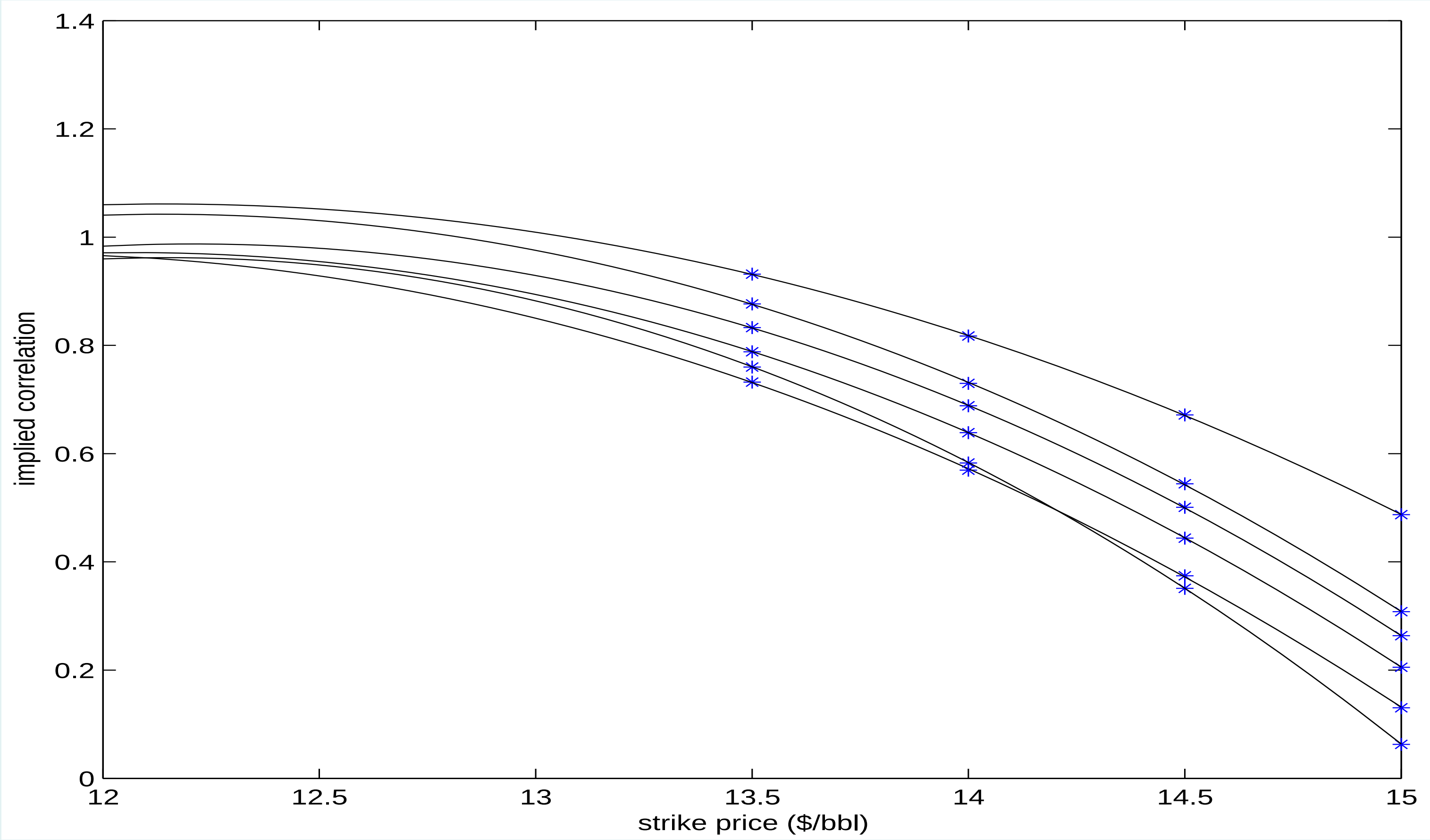
Implied correlations from spreads

Basket option price formula can be *inverted* to obtain implied correlation
Volatilities can be implied from individual assets' options, e.g. ATM.

Correlations implied from NYMEX Brent crude oil/heating oil Asian spread options in 2009 (ATM implied vols for Brent and HO were used):

Strike	Oct. 12	Oct. 13	Oct. 16	Oct. 17	Oct. 18	Oct. 19
15	0.15	0.21	0.48	0.26	0.31	0.10
14.5	0.37	0.44	0.67	0.51	0.54	0.35
14	0.57	0.64	0.82	0.69	0.73	0.59
13.5	***0.73***	***0.79***	***0.93***	***0.83***	***0.88***	***0.76***
12	0.96	0.97	1	0.98	1	0.96

Implied correlations vs strikes



Conclusions

Our proposed method:

- Has advantages of lognormal approximation
- Applicable to several assets, negative weights and Asian basket options
- Provides good approximation of option prices
- Gives closed-form expressions for the greeks
- Performs well on the basis of delta-hedging
- Allows to imply correlations from liquid spread options