Asian basket options and implied correlations in oil markets

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• **Basket option**: option whose underlying is a basket (i.e. a portfolio) of assets.

• Particular case of a basket option: *spread* option

• Payoff of a European basket call option: \((B(T) - X)^+\)
  
  - \(B(T)\) is the basket value at the time of maturity \(T\),
  - \(X\) is the strike price.

• Payoff of an *Asian basket call option*:
  
  - \(B(T)\) is replaced by \(A(T)\): the *average basket value* between times 0 and \(T\).
Commodity baskets and spreads

• Crack spread:

\[ k_u \times \text{Unleaded gasoline} + k_h \times \text{Heating oil} - \text{Crude} \]

• Energy company portfolios:

\[ k_1 \times E_1 + k_2 \times E_2 + \ldots + k_n \times E_n \]

where \( k_i \)'s can be positive as well as negative (i.e. a portfolio can contain both long and short positions).
Motivation:

• Commodity portfolios contain two or more assets, and often contain both *long and short positions*.

• The valuation and hedging of options on such portfolios (i.e. *basket options*) is challenging because *the sum of lognormal r.v.’s is not lognormal*.

• Such portfolios can have negative values, so *lognormal distribution cannot be used, even in approximation*.

• Most existing approaches can only deal with regular basket options or options on a spread between two assets (*Kemna and Vorst, Kirk, Turnbull and Wakeman, …*).

• Numerical and Monte Carlo methods are slow, do not provide closed formulae.

• Need to *extend* pricing and hedging to *Asian basket options*. 
GLN (Generalized Lognormal) approach:

• Essentially a *moment-matching method*.
• Portfolio (i.e. basket) distribution is approximated using a *generalized family of lognormal distributions*:

  *shifted or negative shifted lognormal distribution*

The main attractions:

• applicable to options on portfolio with several long and short positions
• naturally extendable to Asian-style options
• allows to apply Black-Scholes formula
• provides closed formulae for the option price and the greeks
Regular lognormal, shifted lognormal and negative regular lognormal
Assumptions:

- Portfolio consists of futures on different (but related) commodities. The portfolio’s value at time of option maturity $T$

$$B(T) = \sum_{i=1}^{N} a_i F_i(T)$$

where $a_i$ : the weight of asset (futures contract) $i$,
$N$ : the number of assets in the portfolio,
$F_i(T)$: the futures price $i$ at the time of maturity.

- The futures in the portfolio and the option on it mature on the same date.
Individual assets’ dynamics:

Under the risk adjusted probability measure $Q$, the futures prices are martingales. The stochastic differential equations for $F_i(t)$ is

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i . dW^{(i)}(t), i = 1, 2, 3, ..., N$$

where

- $F_i(t)$: the futures price of asset $i$ at time $t$
- $\sigma_i$: the volatility of asset $i$
- $W^{(i)}(t), W^{(j)}(t)$: the Brownian motions driving assets $i$ and $j$ with correlation $\rho_{i,j}$
The first three moments and the skewness of the basket on maturity date $T$ can be calculated:

$$E B(T) = M_1(T) = \sum_{i=1}^{N} a_i . F_i(0)$$

$$E (B(T))^2 = M_2(T) = \sum_{j=1}^{N} \sum_{i=1}^{N} a_i . a_j . F_i(0) . F_j(0) . \exp(\rho_{i,j} . \sigma_i . \sigma_j . T)$$

$$E (B(T))^3 = M_3(T) =$$

$$= \sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{i=1}^{N} a_i . a_j . a_k . F_i(0) . F_j(0) . F_k(0) . \exp(\rho_{i,j} . \sigma_i . \sigma_j . T + \rho_{i,k} . \sigma_i . \sigma_k . T + \rho_{j,k} . \sigma_j . \sigma_k . T)$$

$$\eta_{B(T)} = \frac{E(B(T) - E(B(T)))^3}{\sigma_{B(T)}^3}$$

where $\sigma_{B(T)}$ : standard deviation of basket at the time $T$
• If we assume the distribution of a basket is shifted lognormal with parameters $m, s, \tau$, the parameters should satisfy non-linear equation system:

$$M_1(T) = \exp\left( m + \frac{1}{2} s^2 \right)$$

$$M_2(T) = \tau^2 + 2.\tau.\exp\left( m + \frac{1}{2} s^2 \right) + \exp(2m + 2s^2)$$

$$M_3(T) = \tau^3 + 3.\tau^2.\exp\left( m + \frac{1}{2} s^2 \right) + 3.\tau.\exp(2m + 2s^2) + \exp\left( 3m + \frac{9}{2} s^2 \right)$$

• If we assume the distribution of a basket is negative shifted lognormal, the parameters should satisfy non-linear equation system above by changing $M_1(T)$ to $-M_1(T)$ and $M_3(T)$ to $-M_3(T)$. 
### Approximating distribution:

<table>
<thead>
<tr>
<th>Skewness</th>
<th>$\eta &gt; 0$</th>
<th>$\eta &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximating distribution</td>
<td>shifted</td>
<td>negative shifted</td>
</tr>
</tbody>
</table>
Examples of terminal basket value distribution:

\[ Fo = [100;90]; \sigma = [0.2;0.3]; a = [-1;1]; X = -10; r = 3\%; T = 1 \text{ year}; \rho = 0.9 \]

**Shifted lognormal**

\[ Fo = [105;100]; \sigma = [0.3;0.2]; a = [-1;1]; X = -5; r = 3\%; T = 1 \text{ year}; \rho = 0.9 \]

**Negative shifted lognormal**
Valuation of a European call option (shifted lognormal):

• Suppose that the distribution of basket 1 is lognormal. Then an option on such a basket can be valued by applying the Black-Scholes (actually Black’s (1975)) formula.

• Suppose that the relationship between basket 2 and basket 1 is

\[ B^{(2)}(t) = B^{(1)}(t) + \tau \]

• The payoff of a call option on basket 2 with the strike price \( X \) is:

\[ (B^{(2)}(T) - X)^+ = ((B^{(1)}(T) + \tau) - X) = (B^{(1)}(T) - (X - \tau))^+ \]

It is the payoff of a call option on basket 1 with the strike price \((X - \tau)\)
Valuation of call option (negative lognormal):

• Suppose again that the distribution of basket 1 is lognormal (an option on such a basket can be valued by applying the Black-Scholes formula).

• Suppose that the relationship between basket 2 and basket 1 is

\[ B^{(2)}(t) = -B^{(1)}(t) \]

• The payoff of a call option on basket 2 with the strike price \( X \) is:

\[ \left( B^{(2)}(T) - X \right)^+ = \left( -B^{(1)}(T) - X \right) = \left( (-X) - B^{(1)}(T) \right)^+ \]

It is the payoff of a put option on basket 1 with the strike price \(-X\).
Closed form formulae for a (European) basket call option:

- For e.g. shifted lognormal:

\[
c = \exp(-rT)[(M_1(T) - \tau)N(d_1) - (X - \tau)N(d_2)]
\]

where

\[
d_1 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) + \frac{1}{2}V^2}{V}
\]

\[
d_2 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) - \frac{1}{2}V^2}{V}
\]

\[
V = \sqrt{\log\left(\frac{M_2(T) - 2\tau M_1(T) + \tau^2}{(M_1(T) - \tau)^2}\right)}
\]

It is the call option price with strike price \((X - \tau)\).

Differentiate it w.r.t. parameters \(\rightarrow\) analytic expressions for the greeks
Asian baskets

Underlying: average basket value over a certain interval

Note:

\[ A_B(T) = \frac{1}{n} \sum_{k=t_1}^{t_n} B(t_k) = \frac{1}{n} \sum_{k=t_1}^{t_n} \sum_{i=1}^{N} a_i F_i(t_k) = \sum_{i=1}^{N} a_i A_i(T) \]

So the average basket value is simply the basket of individual assets’ averages, with the same weights

→ assets’ averages are approximated by lognormal distributions, by matching first two moments (as in Wakeman method)

→ the GLN approach then applies directly, only with different moments (calculated from the moments of the average asset prices)

→ closed-form expressions for option prices and greeks.
<table>
<thead>
<tr>
<th></th>
<th>Basket 1</th>
<th>Basket 2</th>
<th>Basket 3</th>
<th>Basket 4</th>
<th>Basket 5</th>
<th>Basket 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Futures price ( (F_0) )</td>
<td>[100;120]</td>
<td>[150;100]</td>
<td>[110;90]</td>
<td>[200;60]</td>
<td>[95;90;105]</td>
<td>[100;90;95]</td>
</tr>
<tr>
<td>Volatility ( (\sigma) )</td>
<td>[0.2;0.3]</td>
<td>[0.3;0.2]</td>
<td>[0.3;0.2]</td>
<td>[0.3;0.2]</td>
<td>[0.2;0.3;0.25]</td>
<td>[0.25;0.3;0.2]</td>
</tr>
<tr>
<td>Weights ( (a) )</td>
<td>[-1;1]</td>
<td>[-1;1]</td>
<td>[0.7;0.3]</td>
<td>[-1;1]</td>
<td>[1; -0.8; -0.5]</td>
<td>[0.6;0.8; -1]</td>
</tr>
<tr>
<td>Correlation ( (\rho) )</td>
<td>0.9</td>
<td>0.3</td>
<td>0.9</td>
<td>0.9</td>
<td>( \rho_{1,2} = \rho_{2,3} = 0.9 )</td>
<td>( \rho_{1,2} = \rho_{2,3} = 0.9 )</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \rho_{1,3} = 0.8 )</td>
<td>( \rho_{1,3} = 0.8 )</td>
</tr>
<tr>
<td>Strike price ( (X) )</td>
<td>20</td>
<td>-50</td>
<td>104</td>
<td>-140</td>
<td>-30</td>
<td>35</td>
</tr>
<tr>
<td>skewsness ( (\eta) )</td>
<td>( \eta &gt; 0 )</td>
<td>( \eta &lt; 0 )</td>
<td>( \eta &gt; 0 )</td>
<td>( \eta &lt; 0 )</td>
<td>( \eta &lt; 0 )</td>
<td>( \eta &gt; 0 )</td>
</tr>
<tr>
<td>Location parameter ( (\tau) )</td>
<td>( \tau &lt; 0 )</td>
<td>( \tau &lt; 0 )</td>
<td>( \tau &gt; 0 )</td>
<td>( \tau &gt; 0 )</td>
<td>( \tau &lt; 0 )</td>
<td>( \tau &lt; 0 )</td>
</tr>
</tbody>
</table>

\( T = 1 \) year; \( r = 3\% \)
Simulation results (Asian call option prices)

<table>
<thead>
<tr>
<th>Method</th>
<th>Basket 1</th>
<th>Basket 2</th>
<th>Basket 3</th>
<th>Basket 4</th>
<th>Basket 5</th>
<th>Basket 6</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GLN</strong></td>
<td>5.9</td>
<td>12.8</td>
<td>8.25</td>
<td>14.5</td>
<td>5.95</td>
<td>7.1</td>
</tr>
<tr>
<td></td>
<td>neg.</td>
<td>neg.</td>
<td>shifted</td>
<td>neg.</td>
<td>neg.</td>
<td>neg.</td>
</tr>
<tr>
<td></td>
<td>shifted</td>
<td>shifted</td>
<td></td>
<td>shifted</td>
<td>shifted</td>
<td></td>
</tr>
</tbody>
</table>

| Monte Carlo    | 5.9      | 12.75    | 8.22     | 14.5     | 5.95     | 7.06     |
|                | (0.11)   | (0.02)   | (0.01)   | (0.02)   | (0.02)   | (0.03)   |
Greeks: correlation vega

Spread [100,120], vols=[0.2,0.3]
Volatility vegas vs. volatilities

same spread, X=20, correlation=0.9
Implied correlations from spreads

Basket option price formula can be *inverted* to obtain implied correlation.
Volatilities can be implied from individual assets’ options, e.g. ATM.

Correlations implied from NYMEX Brent crude oil/heating oil Asian spread options in 2009 (ATM implied vols for Brent and HO were used):

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</thead>
<tbody>
<tr>
<td>15</td>
<td>0.15</td>
<td>0.21</td>
<td>0.48</td>
<td>0.26</td>
<td>0.31</td>
<td>0.10</td>
</tr>
<tr>
<td>14.5</td>
<td>0.37</td>
<td>0.44</td>
<td>0.67</td>
<td>0.51</td>
<td>0.54</td>
<td>0.35</td>
</tr>
<tr>
<td>14</td>
<td>0.57</td>
<td>0.64</td>
<td>0.82</td>
<td>0.69</td>
<td>0.73</td>
<td>0.59</td>
</tr>
<tr>
<td>13.5</td>
<td>0.73</td>
<td>0.79</td>
<td>0.93</td>
<td>0.83</td>
<td>0.88</td>
<td>0.76</td>
</tr>
<tr>
<td>12</td>
<td>0.96</td>
<td>0.97</td>
<td>1</td>
<td>0.98</td>
<td>1</td>
<td>0.96</td>
</tr>
</tbody>
</table>
Implied correlations vs strikes

![Graph showing implied correlations vs strike prices](image-url)
Conclusions

Our proposed method:

- Has advantages of lognormal approximation
- Applicable to several assets, negative weights and Asian basket options
- Provides good approximation of option prices
- Gives closed-form expressions for the greeks
- Performs well on the basis of delta-hedging
- Allows to imply correlations from liquid spread options