Sufficient and necessary conditions for multi-asset optimal stopping

Applications to power plant investment options

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Outline

The plant investment problem

1. At time s, a power plant is characterized by two economic functions
   1.1 its cash flow $\pi(X_s(\omega), s)$
   1.2 its investment (purchase) cost $I(X_s(\omega))$

   that are both functions of the stochastic vector $X_s(\omega) : \Omega \times \Re_+ \rightarrow \Re^n$, $n > 1$ of explanatory variables (the prices).
2. For a power plant, X encompasses typically : the power price ; the fuels, emission, investment and operation costs.
3. Importantly, our problem aims at dealing with several uncertainties ($n > 1$) which may have different natures.
   Mathematical modeling : $X(t, \omega)$ is a very general diffusion. Some of its components may be geometric brownian motions (GBMs) ; others geometric mean reverting processes (GMRPs) or Schwartz processes. . .

Problem formulation (1)

In our problem formulation, the optimal investment in a power plant is the successive solution of Problem 1...

Problem 1 (Determine the NPV of the plant)

$$\begin{aligned}\mathrm{NPV}(x) &= \mathbb{E}\left[\int_0^\infty \pi\left(X_s^x(\omega), s\right) e^{-\rho s} ds - I(X_0^x) \middle| \mathcal{F}_0\right] \\ &= \mathbb{E}^x\left[\int_0^\infty \pi\left(X_s(\omega), s\right) e^{-\rho s} ds\right] - I(x). \end{aligned} \tag{1}$$

Notation :

- $X_s^x(\omega)$ diffusion "X" starting in x at time $s = 0$ (i.e. $X_0^x = x$).
- $\mathbb{E}^x\left[f(X_s)\right] \triangleq \mathbb{E}[f(X_s^x)|\mathcal{F}_0] = \mathbb{E}[f(X_s)|X_0 = x]$.

Problem formulation (2)

. . . and Problem 2.

Problem 2 (Value of the project)

When is the right time to invest in order to receive $\mathrm{NPV}(x)$ *?*

$$\mathrm{V}(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-\rho\tau} \mathrm{NPV}\left(X_\tau\right) \right] \qquad (2)$$

where $\mathcal{S}$ *is the set of (non anticipative and strict) stopping times.*

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On the NPV calculation

The NPV calculation is generally impossible.

Example 1

1. *Take* $X \in \Re^3_+$ *with* $X_1 \sim \mathrm{GMRP}$, $X_2 \sim \textsc{Schw}$, *and* $X_3 \sim \mathrm{GBM}$.
2. *Assume* $\pi(X) = \max(X_1 - X_2 - X_3, 0)$. *It is impossible to solve analytically Problem 1.*

One is forced to use Monte Carlo simulations (always applicable)

On optimal stopping problems (1)

Optimal stopping problems are not analytically solvable with multiple uncertainties.

Example 2 (Perpetual exchange option on GBMs)

Take $X : \Omega \to \Re^{n+m}$ with n, $m \geq 1$. Consider the optimal stopping problem :

$$\tau^\star(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[e^{-\rho\tau} \left(\sum_{i=1}^{n} X_i^\tau - \sum_{j=n+1}^{m} X_j^\tau \right) \right] \tag{3}$$

where X is an $n+m$ dimensional GBM.

1. *This is the "simplest" problem one can think of : all assets are GBMs and the reward function is linear ;*
2. *This problem has no analytically determinable optimal stopping rule for the time being ; except in the case $n = m = 1$, See [5] ;*
3. *But there exists sufficient and necessary conditions for optimal stopping ; See [7], [3] and [6].*

On optimal stopping problems (2)

Optimal stopping problems are not analytically solvable with multiple uncertainties.

Example 3 (Perpetual exchange option on a diffusion mix)

Take $X : \Omega \rightarrow \Re^{n+m}$ *with* n, $m \geq 1$. *Consider the optimal stopping problem :*

$$\tau^{\star}(x,\omega) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x}\left[e^{-\rho\tau}\left(\sum_{i=1}^{n} X_i^{\tau} - \sum_{j=n+1}^{m} X_j^{\tau}\right)\right] \qquad (4)$$

where X *is an* $n+m$ *general diffusion.*

1. X *may mix GBMs, GMRPs, Schwartz processes. . .*
2. *There is no analytically determinable optimal stopping rule for this problem.*
3. *But sufficient and necessary conditions for optimal stopping exist ; See Gahungu and Smeers [2].*

On optimal stopping problems (3)

Optimal stopping problems are not analytically solvable with multiple uncertainties.

1. Which numerical methods can we alternatively use ? In general, backward dynamic programming
   1.1 (binomial/trinomial) Trees
   1.2 backward Monte Carlo computations (Longstaff and Schwartz [4])
2. These methods suffer from
   2.1 the curse of dimensionality
   2.2 inefficiency in parallel computing

On our ability to solve Problems 1 and 2 (Summary)

NPV determination :

1. Is not analytically solvable
2. In practice, requires Monte Carlo

Optimal stopping :

1. Is not analytically solvable for multi asset problems
2. One can only hope for sufficient / necessary conditions for optimal stopping
3. Numerical resolution is impossible for multiple uncertainty problems (typically, $n \geq 4$).

On our ability to solve Problems 1 and 2 (Summary)

1. We can not currently rely on purely analytic or numerical methods to solve Problems 1 and 2 with multiple uncertainties
2. Can we find an efficient semi-analytic approach ? We can try :
   2.1 To compute the NPV by Monte Carlo simulations ;
   2.2 To determine sufficient and necessary conditions for optimal stopping on a regression of the computed NPV.

Outline

1 - NPV calculation : we use Monte Carlo methods

Take $x \in \Re^n$, $\pi : \Re^n \to \Re$. By Monte Carlo method, obtain an estimator $\hat{\mathrm{NPV}}(x)$ of NPV(x).

1. Disadvantage : Monte Carlo simulations are carried out on a n-dimensional grid and the total number of points in a grid suffer from the curse of dimensionality (it is an exponential function of n)
2. But : the procedure is embarrassingly parallel : it allows efficient parallel computing in clusters.
3. Advantage : flexible in the modeling of the profit function $\pi(X)$.

2 - Intermediate step : regression of $\mathrm{N\hat{P}V}(x)$

Regress $\mathrm{N\hat{P}V}(x)$

1. A regression of $\mathrm{N\hat{P}V}(x)$, $x \in \Re^n$.

$$\mathrm{N\hat{P}V}(x) \approx \mathrm{R{\scriptsize EG}\_NPV}(x) \triangleq \sum_{i=1}^{n} \left(\sum_{k=1}^{m_i} c_{ik} f_{ik}\left(x_i\right) \right) \tag{5}$$

2. For all $i = 1, \cdots n$, $\{f_{ik}\}_{k=1,\cdots,m_i}$ is a regression base on the variable x_i.
3. The choice of the regression bases should allow the determination of sufficient and necessary conditions for optimal stopping of

$$\tau^\star(x,\omega) = \arg\sup_{\tau\in\mathcal{S}} \mathbb{E}^x \left[e^{-\rho\tau} \mathrm{R{\scriptsize EG}\_NPV}(X_\tau) \right]. \tag{6}$$

2 - Intermediary step : regression of $\mathrm{N\hat{P}V}(x)$

Regress $\mathrm{N\hat{P}V}(x)$

1. There is large variety of possible regression schemes.
2. However, keeping in mind our subsequent task of determining sufficient and necessary conditions for optimal stopping, the basic polynomial regression scheme turns out to be the most useful.

Definition 1 (The polynomial regression scheme)

For the state variable $X_t(\omega) : \Omega \times \Re_+ \to \Re^n$, define the regression scheme

$$\mathrm{P{\scriptstyle OL}\_NPV}(x) \triangleq \sum_{i=1}^{n} 1_i \left(\sum_{k=1}^{m_i} c_{ik} x_i^{\alpha_{ik}} \right) \tag{7}$$

with $1_i = +1$ (resp. $1_i = -1$) if asset i is a price (resp. cost), $c_{ik} \geq 0$ $\forall i, k$.

3 - Sufficient and necessary conditions for multi-asset optimal stopping

Determine sufficient and necessary conditions for optimal stopping

It remains to know

1. for which diffusions in X
2. under which conditions on α_{ik}

the regression model (7) allows to characterize (via sufficient or necessary conditions for optimal stopping) the stopping region of

$$\tau^{\star}(x) = \arg\sup_{\tau \in \mathcal{S}} \mathbb{E}^{x} \left[\mathrm{POL\_NPV}\left(X_{\tau} \right) \right]. \tag{8}$$

3 - Sufficient and necessary conditions for multi-asset optimal stopping

Determine sufficient and necessary conditions for optimal stopping

We find (See [1]) that X may contains

1. $X_i \sim$ GBM under the conditions $0 \leq \alpha_{ik} \leq \gamma_+$ where γ_+ is the positive root of a quadratic form ;
2. $X_i \sim$ Schwartz process under the conditions $0 \leq \alpha_{ik} \leq 1$
3. $X_i \sim$ SBM with drift under the conditions $0 \leq \alpha_{ik} \leq 1$ if 1_i=1, $\alpha_{ik} \geq 1$ if 1_i=-1.
4. if $X_i \sim$ GMRP ; the conditions are hard to determine.

Outline

Example 1

1. $X \in \Re^4_+$.
2. $\pi : \Re^3 \rightarrow \Re$.

$$\pi(X) = \max\left(X_1 - X_2 - X_3, 0\right) \tag{9}$$

   "The plant has the option to costlessly shut down if the spark spread is negative."
3. $I(X) = X_4$. "The investment cost is uncertain"
4. Uncertainty :
   $X_1 \sim$ GMRP(0.02,0.2,50), $X_2 \sim$ Schw(0.03,0.3,ln(37)), $X_3 \sim$ GBM(0.02,0.2) and $X_4 \sim$ GBM(0.03,0.25) ; the discount rate $\rho = 0.1$
5. Monte Carlo : Numerical computation of the NPV on an horizon 50 years using quarterly average prices. Monte Carlo worked out on a mesh of initial values of $6 \times 8 \times 15 = 720$ points, using $N = 500$ events per point. Required time : around 5 minutes (on a MacBook pro 2.8GHz).

7. Regression by

$$\textsc{Pol\_NPV}(x) \triangleq \sum_{i=1}^{3} 1_i \left(\sum_{k=1}^{100} c_{ik} x_i^{\alpha_{ik}} \right) - X_4 \qquad (10)$$

with $1_1 = -1_2 = -1_3 = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0.01 : 0.01 : 1$. We were thus looking for 300 positive coefficients. We used the function *lsqnonneg* in Matlab 2009b.

   a. Required time : around 2 minutes (on a MacBook pro 2.8GHz).
   b. Relative regression error : 3.53%.

8. A sufficient condition for optimal stopping takes the form

$$\begin{aligned} C^{-1}_{\mathrm{REG\_GMRP}(c_1)}(X_1) \quad \geq \quad & P^{-1}_{\mathrm{REG\_SCHW}(c_2)}(X_2) + P^{-1}_{\mathrm{REG\_GBM}(c_3)}(X_3) \\ + \quad & P^{-1}_{\mathrm{GBM}}(X_4) \end{aligned}$$

where the trigger functions $C_{\mathrm{REG\_GMRP}}$, $P_{\mathrm{REG\_SCHW}}$, $P_{\mathrm{REG\_GBM}}$, P_{GBM} are invertible. See Gahungu and Smeers [1], Appendix B.

Outline

Conclusion

1. In a multi asset environment, both analytic NPV calculation and resolution of optimal stopping problems are impossible ;
2. But one can always use Monte Carlo simulations for the NPV computation ;
3. And one can often compute (analytically) sufficient and necessary conditions for optimal stopping for polynomial regression schemes of NPV.

Outline

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