# Forward pricing in electricity markets based on stable CARMA spot models

Fred Espen Benth, Claudia Klüppelberg and Linda Vos

Center of Mathematics for applications (CMA)
University of Oslo

Faculty of economics and social sciences
University of Agder

Center of Mathematical Sciences and Institute of Advanced Study
Munich University of Technology

# Outline

# What has been done...

- Bernhardt, Klüppelberg and Meyer-Brandis [1] found $\alpha$-Stable ARMA model good model for spot.
- Garcia *et. al* [3] generalized this approach to $\alpha$-Stable continuous time ARMA (CARMA) models.

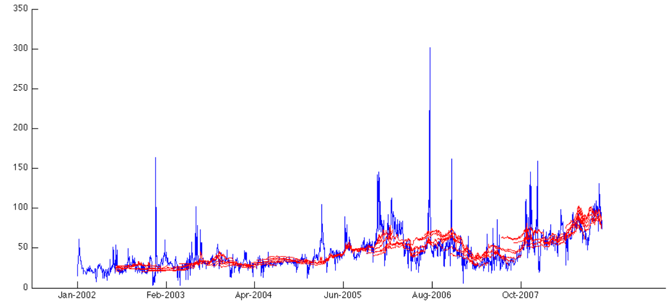
# Aim of this project

- Modeling of Forward dynamics in energy market with "short rate" model.
- Based on no-arbitrage arguments the forward dynamics is given by

$$f(t,\tau) = \mathbb{E}_Q[S(\tau)|\mathcal{F}_t]$$

- Describe **relation** between spot and forward.

# Data curves


350
300
250
200
150
100
50
0
Jan-2002
Feb-2003
Apr-2004
Jun-2005
Aug-2006
Oct-2007

# $p$-dimensional arithmetic model

Spot given by sum OU-processes.

$$S(t) = \Lambda(t) + \sum_{i=1}^{p} X_i(t)$$

where $X_i$ are OU-processes.

$$dX_i(t) = -\eta_i X_i(t)dt + \sigma_i dL_i(t)$$

By using

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \mathbb{E}_Q[\int_{T_1}^{T_2} S(\tau)d\tau | \mathcal{F}_t]$$

# $p$-dimensional arithmetic model

Spot given by sum OU-processes.

$$S(t) = \Lambda(t) + \sum_{i=1}^{p} X_i(t)$$

where $X_i$ are OU-processes.

$$dX_i(t) = -\eta_i X_i(t)dt + \sigma_i dL_i(t)$$

By using

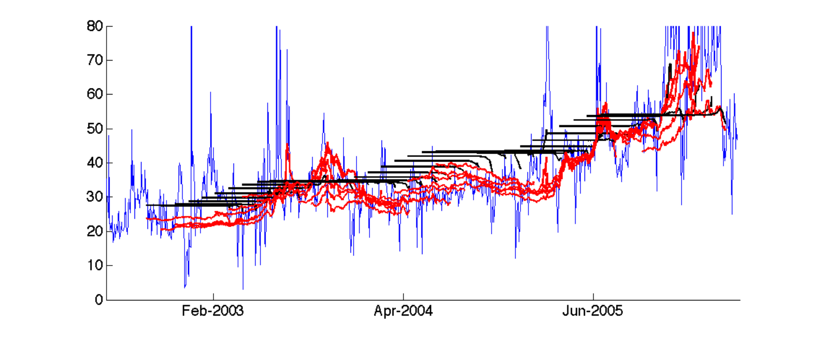
$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \mathbb{E}_Q[\int_{T_1}^{T_2} S(\tau)d\tau | \mathcal{F}_t]$$

# The forward dynamics

It holds

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau$$
$$+ \frac{1}{T_2 - T_1} \sum_i -\frac{1}{\eta_i} \left( e^{-\eta_i (T_2 - t)} - e^{-\eta_i (T_1 - t)} \right) X_i(t)$$
$$+ \frac{1}{\eta_i} \sigma_i \mathbb{E}_Q[L_i(1)] + \frac{1}{\eta_i^2} \left( e^{-\eta_i (T_2 - t)} - e^{-\eta_i (T_1 - t)} \right) \sigma_i \mathbb{E}_Q[L_i(1)]$$

# Dependence on Seasonality function



# Our spot model

The electricity spot $S$ is given by

$$S(t) = \Lambda(t) + Y(t) + Z(t)$$

- $\Lambda$ is a seasonality function.
- $Y$ is an $\alpha$-stable CARMA process.
- $Z$ is a non-stationary term modeled by a NIG process.

# Seasonality function

The seasonality is given by

$$\Lambda(t) = c_0 + c_1 t + c_2 \sin\left(\frac{2\pi t}{365}\right) + c_3 \cos\left(\frac{2\pi t}{365}\right)$$

# $\alpha$-stable distribution

Advantage $\alpha$-stable distribution.

- 4 parameter distribution $\alpha, \beta, \gamma, \mu$.
- Closed under addition. $x_1 \sim S_{\alpha_1}(\beta_1, \gamma_1, \mu_1)$ and $x_2 \sim S_{\alpha_2}(\beta_2, \gamma_2, \mu_2)$ the $x_1 + x_2$ is again stable.
- Permits extreme heavy tails which are observed in electricity prices.
- Brownian motion is a special case of this distribution.

Disadvantage $\alpha$-stable distribution.

- Has only moments up to $\alpha$.

# $\alpha$-stable distribution

Advantage $\alpha$-stable distribution.

- 4 parameter distribution $\alpha, \beta, \gamma, \mu$.
- Closed under addition. $x_1 \sim S_{\alpha_1}(\beta_1, \gamma_1, \mu_1)$ and $x_2 \sim S_{\alpha_2}(\beta_2, \gamma_2, \mu_2)$ the $x_1 + x_2$ is again stable.
- Permits extreme heavy tails which are observed in electricity prices.
- Brownian motion is a special case of this distribution.

Disadvantage $\alpha$-stable distribution.

- Has only moments up to $\alpha$.

# CARMA-process

Continuous version of ARMA(p,q) process $\{x_n\}_{n\geq 0}$

$$x_n = \phi_1\, x_{n-1} + \phi_2\, x_{n-2} + \ldots + x_{n-p} + \epsilon_n + \epsilon_{n-1} + \ldots + \epsilon_{n-q}$$

$\{\epsilon_n\}_{n\geq 0}$ is noise.

# Ornstein-Uhlenbeck process

AR(1,0) process $x_n$ has innovations

$$x_n = \phi_1 x_{n-1} + \epsilon_n$$

The continuous variant a CAR(1,0) process $X$ has innovations

$$X(t) = e^{a(t-s)} X(s) + \int_s^t e^{a(t-u)} dB(u)$$

# Definition CARMA-process

Let us introduce a CARMA($p, q$)-Lévy process (see Brockwell [2]):

## Theorem (CARMA($p, q$)-Lévy process)

*A CARMA(p,q)-Lévy process $\{Y(t)\}_{t\geq 0}$ (with $0 \leq q < p$) is defined to be a stationary solution of the p-th order differential equation,*

$$a(D)Y(t) = b(D)DL(t), \quad t \geq 0 \tag{1}$$

*where*

$$a(z) := z^p + a_1 z^{p-1} + \ldots + a_p$$
$$b(z) := b_0 + b_1 z + \ldots + b_q z^q$$

*are the characteristic polynomials with real coefficients $a_1, \ldots, a_p, b_0, \ldots, b_q$. The notation $D$ indicates a differential operator with respect to $t$. It is assumed that $b_q = 1$ and we define $b_j := 0$ for $q < j \leq p$.*

# $\alpha$-Stable CARMA process

An $\alpha$-stable CARMA(p,q) process $Y$ has a state-space representation given by

$$Y(t) = \mathbf{b}^* X(t)$$

$$X(t) = e^{A(t-s)}X(s) + \int_s^t e^{A(t-u)}\mathbf{e}_p dL(u)$$

- $\mathbf{b}^*$ is a $1 \times p$ parameter vector.
- $A$ is a $p \times p$ matrix with the autoregressive parameters as its parameters.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \vdots & -a_1 \end{pmatrix}.$$

- $\mathbf{e}_p$ is a $p \times 1$ unit vector with a 1 on the $p$-th entry.

# Model; The forward

From no-arbitrage arguments we have that

$$F(t, T_1, T_2) = \mathbb{E}_Q\left[\int_{T_1}^{T_2} w(u, T_1, T_2) S(u) du \middle| \mathcal{F}_t\right]$$

$w$ is a weight function,

$$w(u, T_1, T_2) = \frac{e^{-ru}}{e^{-rT_2} - e^{-rT_1}} \quad \text{or} \quad w(u, T_1, T_2) = \frac{1}{T_2 - T_1}$$

where $r$ is the interest-rate.

$$
\begin{aligned}
F(t, T_1, T_2) = &\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \Lambda(\tau)d\tau + Z(t) \\
&+ \frac{\mathbf{b}^* A^{-1}}{T_2 - T_1}\left(e^{AT_2} - e^{AT_1}\right) e^{-At}\,\mathbf{X}(t) \\
&- \left(\frac{\mathbf{b}^* A^{-2}}{T_2 - T_1}\left(e^{AT_2} - e^{AT_1}\right) e^{-At} - \mathbf{b}^* A^{-1}\right) e_p\,\mathbb{E}_Q[L(1)] \\
&+ \left(\frac{1}{2}(T_2 + T_1) - t\right)\mathbb{E}_Q[Z(1)]
\end{aligned}
$$

where

- $\mathbf{X}$ are the states of the CARMA process $Y$.
- $\mathbb{E}_Q[L(1)]$ and $\mathbb{E}_Q[Z(1)]$ are means in risk-neutral world $Q$.

# Equivalent measure

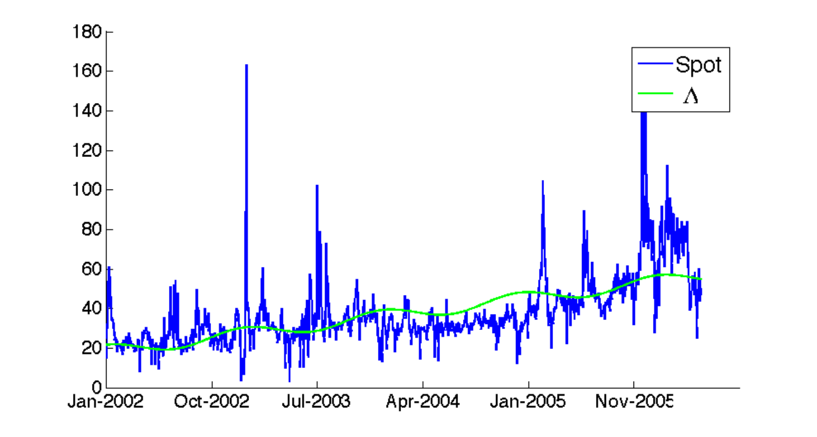
- Equivalent measure is sufficient.
- For non-stationary term $Z$ we use Esscher transform.
- Tempered stable distribution is equivalent to stable.

# Tempered stable distribution

Suppose $\nu_S$ is the Lévy measure of a Lévy process then

$$\nu_{TS}(x) := e^{-\theta x}\nu_S(x)1_{\{x\geq 0\}} + e^{-\theta x}\nu_S(x)1_{\{x\leq 0\}}$$

is Lévy measure of tempered stable distribution for a $\theta \in \mathbb{R}^+$.




Robust least-squares estimate in order to estimate $\Lambda(t)$.

$$
\begin{aligned}
F(t, T_1, T_2) = & \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau + Z(t) \\
& + \frac{\mathbf{b}^* A^{-1}}{T_2 - T_1} \left( e^{AT_2} - e^{AT_1} \right) e^{-At} \mathbf{X}(t) \\
& - \left( \frac{\mathbf{b}^* A^{-2}}{T_2 - T_1} \left( e^{AT_2} - e^{AT_1} \right) e^{-At} - \mathbf{b}^* A^{-1} \right) e_p \, \mathbb{E}_Q[L(1)] \\
& + \left( \frac{1}{2}(T_2 + T_1) - t \right) \mathbb{E}_Q[Z(1)]
\end{aligned}
$$

where

- $\mathbf{X}$ are the states of the CARMA process $Y$.
- $\mathbb{E}_Q[L(1)]$ and $\mathbb{E}_Q[Z(1)]$ are means in risk-neutral world $Q$.

# Far out of maturity

$$F(t, T_1, T_2) \approx \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \Lambda(\tau) d\tau + Z(t)$$
$$+ \mathbf{b}^* A^{-1} e_p \, \mathbb{E}_Q[L(1)] + \left( \frac{1}{2}(T_2 + T_1) - t \right) \mathbb{E}_Q[Z(1)]$$

where

- $\mathbb{E}_Q[L(1)]$ and $\mathbb{E}_Q[Z(1)]$ are means in risk-neutral world $Q$.

# Estimation realization Z

Therefor

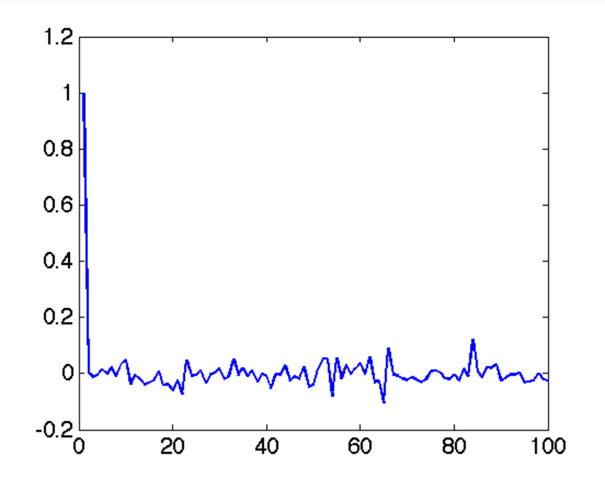
$$\widetilde{F}(t, T_1, T_2) := F(t, T_1, T_2) - \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \Lambda(\tau)d\tau$$
$$\approx Z(t) + \mathbf{b}^* A^{-1}\mathbf{e}_p \, \mathbb{E}_Q[L(1)] + \left(\frac{1}{2}(T_1 + T_2) - t\right) \mathbb{E}_Q[Z(1)].$$

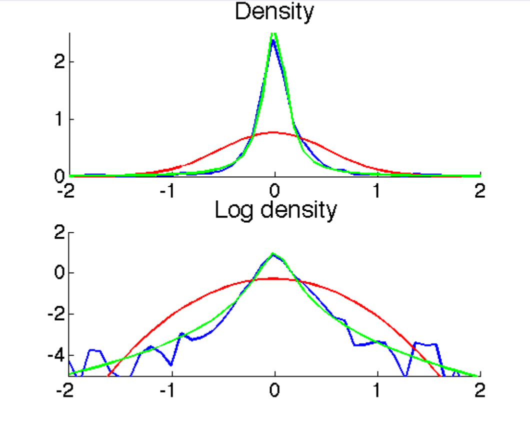
Taking means

$$\mathbb{E}[\widetilde{F}(t, u)] = C + u \, \mathbb{E}_Q[Z(1)]$$

Here $C \in \mathbb{R}$ is a constant $C := \mathbf{b}^* A^{-1}\mathbf{e}_p \, \mathbb{E}_Q[L(1)]$ and $u := \frac{1}{2}(T_1 + T_2) - t$.

# Distribution increments of $Z$






(blue) Kernel density, (red) Gaussian density, (green) NIG density

# Spot dynamics

$$S(t) = \Lambda(t) + Y(t) + Z(t)$$

- $\Lambda$; seasonality function.
- $Y$; $\alpha$-stable CARMA process.
- $Z$; non-stationary NIG process.

# Estimation CARMA(p,q) process for embedable ARMA processes

Based on the following discretization

$$\prod_{i=1}^{p}(1 - e^{\lambda_i} B) y_n = u_n$$

where

$$u_n = \sum_{i=1}^{p} \kappa_i \prod_{j \neq i}(1 - e^{\lambda_j h} B) \int_{(n-1)h}^{nh} e^{\lambda_i (nh-u)} dL(u) . \tag{2}$$

- Roots of

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

equal to $e^{\lambda_i h}$.

- To recover the parameters of $A$, use the fact that the characteristic polynomial $P$ of $A$ is given by

$$P(\lambda) = 1 + a_1\lambda + a_2\lambda^2 + \ldots + a_p\lambda^p$$

For the found eigenvalues $\lambda_i$ it holds that $P(\lambda_i) = 0$. Use standard polynomial techniques to recover $a_i$'s.

- Moving average parameter on empirical auto-correlation function.

- Roots of

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

equal to $e^{\lambda_i h}$.

- To recover the parameters of $A$, use the fact that the characteristic polynomial $P$ of $A$ is given by

$$P(\lambda) = 1 + a_1\lambda + a_2\lambda^2 + \ldots + a_p\lambda^p$$

For the found eigenvalues $\lambda_i$ it holds that $P(\lambda_i) = 0$. Use standard polynomial techniques to recover $a_i$'s.

- Moving average parameter on empirical auto-correlation function.

- Roots of

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

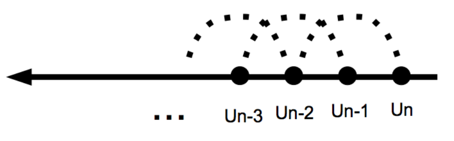
equal to $e^{\lambda_i h}$.

- To recover the parameters of $A$, use the fact that the characteristic polynomial $P$ of $A$ is given by

$$P(\lambda) = 1 + a_1\lambda + a_2\lambda^2 + \ldots + a_p\lambda^p$$

For the found eigenvalues $\lambda_i$ it holds that $P(\lambda_i) = 0$. Use standard polynomial techniques to recover $a_i$'s.

- Moving average parameter on empirical auto-correlation function.




If one looks to the noise then $u_n$ (2) then $p$-steps back dependent. So one can make $p$ series

$$R_1 = \{u_i\}_{(i=1,p+1,2p+1,\ldots)}$$
$$\vdots$$
$$R_p = \{u_i\}_{(i=p,2p,3p,\ldots)}$$

of independent noise and estimate parameters on this.

# $L^1$ filter to recover the states $\mathbf{X}$

We have the following state-space for CARMA processes

$$y_n = \mathbf{b}^* x_n$$
$$x_n = e^{A\Delta} x_{n-1} + z_n$$

here $\Delta$ is step-size.

$$\mathbf{b}^* z_n | y_n, x_{n-1} = y_n - \mathbf{b}^* e^{A\Delta} x_{n-1} \quad (3)$$

# $L^1$ filter to recover the states $\mathbf{X}$

We have the following state-space for CARMA processes

$$y_n = \mathbf{b}^* x_n$$
$$x_n = e^{A\Delta} x_{n-1} + z_n$$

here $\Delta$ is step-size.

$$\mathbf{b}^* z_n | y_n, x_{n-1} = y_n - \mathbf{b}^* e^{A\Delta} x_{n-1} \tag{3}$$

# $L^1$ filter to recover the states $\mathbf{X}$

when uniform effect on $L^n(t) = L(t)1_{\{(n-1)\Delta < t < n\Delta\}}$

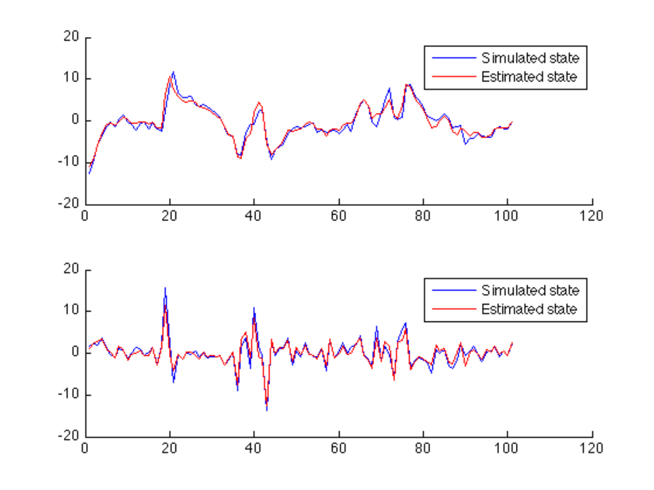
$$\mathbb{E}[z_n | y_n, x_{n-1}] = -A^{-1}\left(I - e^{Ah}\right)\mathbf{e}_p\,\mathbb{E}[L^n(1) \mid y_n, x_{n-1}] \tag{4}$$

combining with (3)

$$\mathbb{E}[L^n(1) \mid y_n, x_{n-1}] = \frac{y_n - \mathbf{b}^* e^{Ah} x_{n-1}}{-\mathbf{b}^* A^{-1}\left(I - e^{Ah}\right)\mathbf{e}_p}$$

plug this back in (4).

# Simulation


Simulated state
Estimated state
20
10
0
-10
-20
0
20
40
60
80
100
120
Simulated state
Estimated state
20
10
0
-10
-20
0
20
40
60
80
100
120

# Risk-premium

The risk-premium is given by

$$R_{pr}(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\mathbb{E}_Q[\int_{T_1}^{T_2} S(\tau)d\tau|\mathcal{F}_t] - \mathbb{E}_P[\int_{T_1}^{T_2} S(\tau)d\tau|\mathcal{F}_t]\right)$$

# Theoretical risk-premium

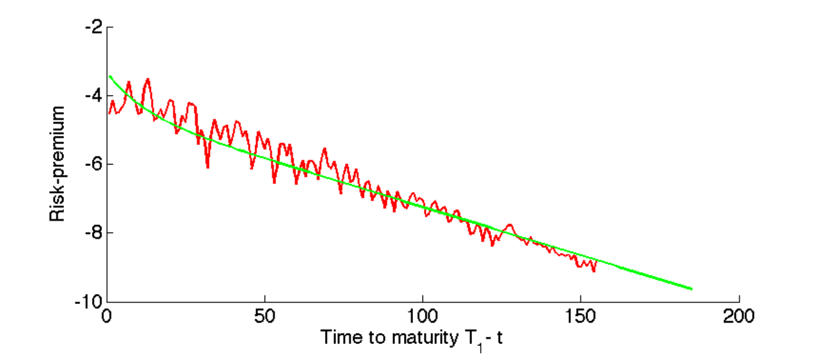
$$
\begin{aligned}
R_{pr}(t, T_1, T_2) = & -\frac{\mathbf{b}^* A^{-2}}{T_2 - T_1} \left( e^{AT_2} - e^{AT_1} \right) e^{-At} \mathbf{e}_p \left( \mathbb{E}_Q[L(1)] - \mathbb{E}_P[L(1)] \right) \\
& + \mathbf{b}^* A^{-1} \mathbf{e}_p \left( \mathbb{E}_Q[L(1)] - \mathbb{E}_P[L(1)] \right) \\
& + \left( \frac{1}{2}(T_1 + T_2) - t \right) \mathbb{E}_Q[Z(1)]
\end{aligned}
$$

# Empirical risk-premium

Take mean of all observed risk-premiums

$$\widetilde{R}_{pr}(u,v) := \frac{1}{v}\mathbf{b}^* A^{-2}\left(e^{\frac{1}{2}Av} - e^{-\frac{1}{2}Av}\right)e^{Au}\mathbf{e}_p\,\mathbb{E}[L(1)] - \mathbf{b}^* A^{-1}\mathbf{e}_p\,\mathbb{E}[L(1)]$$
$$+ \frac{1}{\#U(u,v)}\sum_{t,T_1,T_2\in U(u,v)}\Bigg[F(t,T_1,T_2) - \frac{1}{T_2-T_1}\int_{T_1}^{T_2}\Lambda(\tau)d\tau$$
$$- \frac{\mathbf{b}^* A^{-1}}{T_2-T_1}\left(e^{AT_2} - e^{AT_1}\right)e^{-At}\mathbf{X}(t) - Z(t)\Bigg]$$

# Risk-premium




(green) Theoretical risk premium, (red) mean empirical risk-premium.

# Estimation

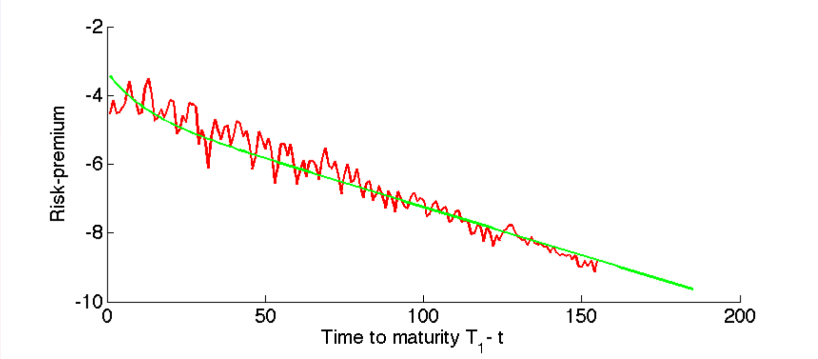
Choose a time far out of maturity $\tilde{u}$

- Estimate the seasonality function $\Lambda$ with robust least squares.
- Estimate non-stationary term $Z$ on Forward curves. With this also risk-premium parameters.
- Estimate CARMA parameters on the residuals using an embedable ARMA process (Brockwell's method).
- Use $L^1$ filter to estimate the states.
- Compare theoretical risk-premium with empirical observed risk-premium.

# Risk-premium comparison

$$\text{argmin}_{\hat{u}} \sum |R_{pr}(u,v) - \widetilde{R}_{pr}(u,v)|^2$$

# Risk-premium




(green) Theoretical risk premium, (red) mean empirical risk-premium.

# Market price of risk

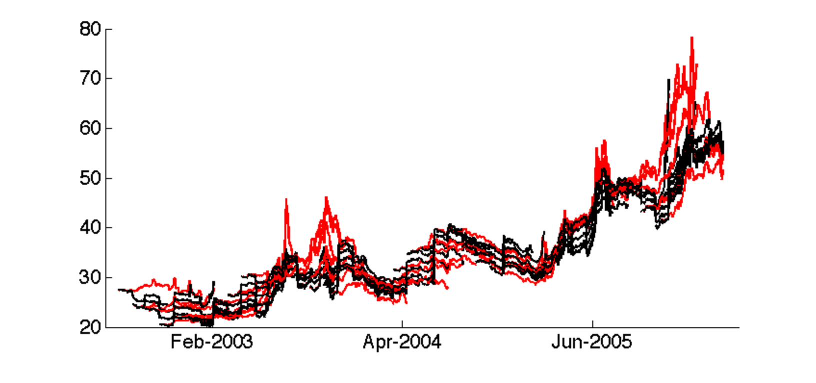
Producers are price makers

$$\frac{1}{T_2 - T_1}\mathbb{E}\left[\int_{T_1}^{T_2} S(\tau)d\tau|\mathcal{F}_t\right] > F(t, T_1, T_2)$$

Since we know values of $\mathbb{E}_Q[L(1)]$ and $\mathbb{E}_Q[Z(1)]$ we can derive market price of risk

$$\theta_Z = -0.0897$$
$$\theta_L = -0.8859$$

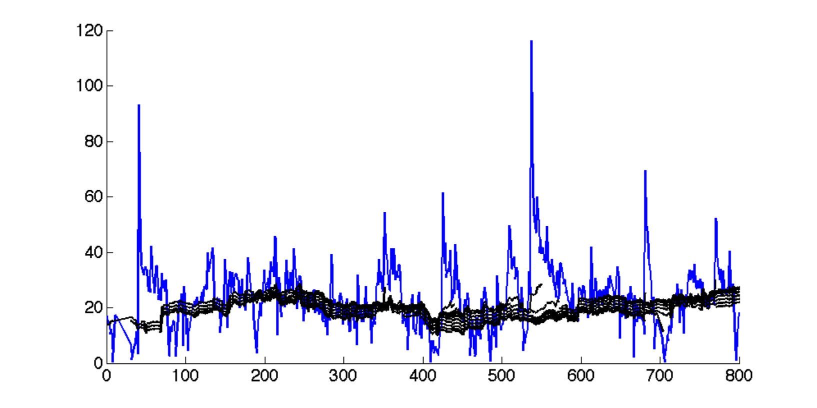
# period Jan-2003-April-2006



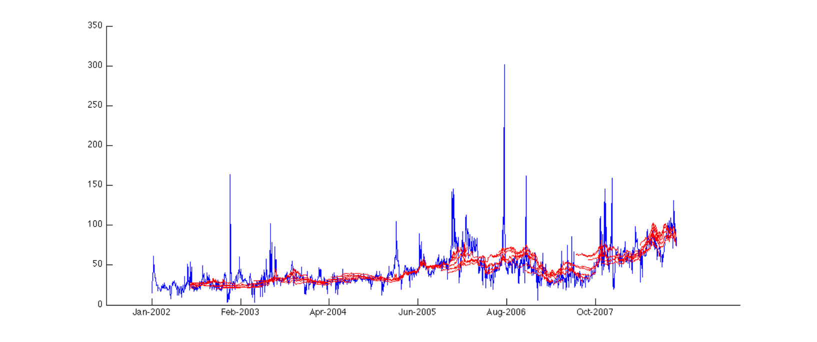

(Black) modeled forward curves, (red) observed forward curves.

# Simulation

A simulation of our model



# Original time-series



# Conclusion:

- We introduced a model for modeling spot and forward dynamics in the electricity market.
- We have taken a non-stationary term for modeling low-frequency dynamics.
- $\alpha$-stable modeling is chosen to model the extreme behavior observed in electricity data.

- An $L^1$ filter is introduced to recover states from a CARMA-process.
- Equivalent measures are described.
- Risk-premium is negative and linearly decaying towards maturity.

# Bibliography


- Bernhardt, C. Klüppelberg, C. Meyer-Brandis, T.(2008). Estimating high quantiles for electricity prices by stable linear models.
- Brockwell, P.J. (2001). Lévy driven CARMA processes. *Ann. Inst. Math.* **53**, pp. 113-124.
- Garcia, I. Klüppelberg, C. and Müller, G. (2009). Estimation of stable CARMA models with an application to electricity spot prices.