Expected Utility and Performance Measures

Aleš Černý

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Goals of the talk

Investors’ preferences
Optimal portfolio selection
Existence of optimal portfolios
Performance measurement
Problems with Sharpe ratio
References
Goals of the talk

- Understand basic paradigms in optimal portfolio selection
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- Expected utility maximization
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  - Standard measurement of attitude to risk
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  - Certainty equivalent growth rate
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  - Sharpe ratio and its shortcomings
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Textbook, chapter 3: Černý 2009

“Aleš Černý’s new edition of Mathematical Techniques in Finance is an excellent master’s-level treatment of mathematical methods used in financial asset pricing. By updating the original edition with methods used in recent research, Černý has once again given us an up-to-date first-class textbook treatment of the subject.”

—Darrell Duffie, Stanford University
Investors’ preferences I

Investors’ preferences

Utility function
Certainty equivalent
Risk aversion

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Assumption #1: Investors prefer more to less
Assumption #2: Investors are risk averse

Definition: An investor is risk averse when positive deviations from her average wealth do not compensate for equally large and equally probable negative deviations.

The two assumptions are captured by a concave and increasing function $U$, commonly called a utility function.
Investors’ preferences I

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Utility function
Utility function

Utility of Wealth $U(V)$ vs Wealth $V$

Concave and increasing utility function
Utility function

Concave and increasing utility function

Utility of Wealth

U(V)

U(V_0)

Wealth V

V_0

V_{up}

V_{down}
Utility function

Concave and increasing utility function

Utility of Wealth

$U(V)$

$U(V_0)$

$V_0$

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$V_{down}$

Wealth $V$

$U(V_{down})$

$U(V_{up})$
Utility function

- Utility function: Concave and increasing
- Utility of Wealth
  - $U(V)$
  - $U(V_0)$
  - $U(V_{up})$
  - $U(V_{down})$
- Wealth $V$
- $V_0$, $V_{up}$, $V_{down}$
Utility function

Concave and increasing utility function

Utility of Wealth

\[ U(V) \]

\[ U(V_{up}) \]

\[ U(V_{0}) \]

\[ V_{down} \]

\[ V_{0} \]

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Wealth V
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Utility function

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- $U(V)$
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- $U(V_{up})$
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- $V_{down}$
- $V_0$
- $V_{up}$
- Wealth $V$

Graph showing concave and increasing utility function.
Examples of utility functions

Examples of utility functions

Two important parametric forms (power and exponential)

CRRA class (Constant Relative Risk Aversion, one parameter)

\[ \text{CRRA}(V) = V^{1/n} \]

CARA utility (Constant Absolute Risk Aversion, one parameter)

\[ \text{CARA}(V) = e^{\alpha V} \text{ with } \alpha > 0 \]

Their generalization

HARA class (Hyperbolic Absolute Risk Aversion, two parameters)

\[ \text{HARA}(V) = \left( V + V_0 \right)^{-1} \]

\[ \text{HARA}(V) = \left( V^j V_0 \right)^{-1} \text{ with } j < 0 \]
Examples of utility functions

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- Their generalization
  - HARA class (Hyperbolic Absolute Risk Aversion, two parameters)
    \[ HARA_{\gamma,\nu}(V) = \frac{(\tilde{V} + V)^{1-\gamma}}{1-\gamma} \text{ with } \gamma > 0, \]
    \[ HARA_{\gamma,\nu}(V) = \frac{|\tilde{V} - V|^{1-\gamma}}{1-\gamma} \text{ with } \gamma < 0 \]
Assumption #3: Risky distribution of wealth is valued by the certainty equivalent of its expected utility:

\[ \text{CE} \text{ its certainty equivalent} \]

\[ U(\text{CE}) = E[U(V)] \]
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$V$ risky distribution of wealth
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Investors’ preferences II

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Expected utility and certainty equivalent wealth
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Performance Measures
A. Černý

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Expected utility and certainty equivalent wealth
An investor who is highly averse to risk will naturally invest less in risky assets.
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Suppose investor’s preferences are generated by a given utility function $U(V)$. 

We call $A(v) := U_0'(v) U_0''(v)$ the coefficient of local absolute risk aversion.
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Measurement of risk aversion I

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Initial wealth is risk-free and equals $v$.

Terminal wealth is $v + \varepsilon$. 
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Terminal wealth is $v + \epsilon$.

Observe how CE varies with $\sigma^2$.

$$CE - v = \frac{1}{2} \frac{U''(v)}{U'(v)} \sigma^2 + o(\sigma^2)$$

We call $A(v) := -\frac{U''(v)}{U'(v)}$ the coefficient of local absolute risk aversion.
Measurement of risk aversion II

- Derivation: write down 2nd order Taylor expansion

\[ U(v + \epsilon) = U(v) + U'(v)\epsilon + \frac{1}{2}U''(v)\epsilon^2 + o(\epsilon^2) \]

- Take expectations on both sides

\[ E[U(v + \epsilon)] = U(v) + \frac{1}{2}U''(v)\sigma^2 + o(\sigma^2) \]

- Write down 1st order expansion for the certainty equivalent

\[ U(CE) = U(v + CE - v) = U(v) + U'(v)(CE - v) + o(CE - v) \]

- From \( U(CE) = E[U(v + \epsilon)] \) we find

\[ CE - v = \frac{1}{2} \frac{U''(v)}{U'(v)} \sigma^2 + o(\sigma^2). \]

- The difference \( CE - v \) is the risk premium
Now assume the shock is multiplicative
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i.e. terminal wealth equals $(1 + \epsilon)V$
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\[
\frac{CE - v}{v} = \frac{1}{2} \frac{vU''(v)}{U'(v)} \sigma^2 + o(\sigma^2)
\]

We call \(R(v) := -\frac{vU''(v)}{U'(v)}\) the coefficient of local relative risk aversion
Basics of optimal portfolio selection

Performance Measures

A. Černý

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By portfolio allocation we mean 2 things:
- allocation of wealth across risky assets
- allocation of wealth between safe and risky assets

In this talk we will study mainly the second aspect.

Optimal portfolio selection is about balancing risk and reward. Mathematically this is achieved by maximizing expected utility of terminal wealth.
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Expected utility maximization - numerical example

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- You wish to invest your savings for a year
- You can invest either in a safe account with a rate of return 2% p.a.
- Or a risky stock, returning either 20% or -10% with equal probability
- **Your task:**
  1. Write down how much (out of your £1 million) you would invest in the stock
  2. Calculate how much a person with utility function $U(V) = -V^{-4}/4$ should invest in the stock
Existence of optimal portfolios I
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- The effective domain of $U$ is the set of points where $U$ is finite.
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We require continuity as we move from inside $\text{dom } U$ to its boundary. Mathematically,

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function $U$ with this property is called closed or upper semi-continuous.
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  \[ \lim_{y \to x} \sup U(y) = U(x) \text{ for all } x \in \mathbb{R}, \]

- function $U$ with this property is called closed or upper semi-continuous

- Example of a discontinuous but closed concave function
  \[ U(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0, \\ -\infty & \text{for } x < 0, \end{cases} \]
Existence of optimal portfolios II

Performance Measures
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Concave functions need not be differentiable at every interior point $x$ of dom $U$ but they always possess left and right derivatives

$$U'_+(x) : = \lim_{h \to 0^+} \frac{U(x + h) - U(x)}{h},$$

$$U'_-(x) : = \lim_{h \to 0^-} \frac{U(x + h) - U(x)}{h}.$$
Concave functions need not be differentiable at every interior point $x$ of $\text{dom } U$ but they always possess left and right derivatives

$$U_+(x) : = \lim_{h \to 0^+} \frac{U(x + h) - U(x)}{h},$$
$$U_-(x) : = \lim_{h \to 0^-} \frac{U(x + h) - U(x)}{h}.$$

Outside the effective domain we set:

$$U_-(x) = U_+(x) = \infty \text{ for } x < \inf \text{ dom } U,$$
$$U_-(x) = U_+(x) = -\infty \text{ for } x > \sup \text{ dom } U.$$
Theorem (Černý et al. 2008)

Suppose $U : \mathbb{R} \to [-\infty, \infty)$ is a closed concave function and there is an open interval $\text{dom}_+ U$ on which $U$ is strictly increasing. Assume

$$\frac{U'(\infty)}{U'(-\infty)} \leq 0,$$

where we adopt the convention $\frac{-\infty}{\infty} \leq 0$.

Let $X$ be an $\mathbb{R}^n$-valued bounded random variable and suppose there exists a probability measure $Q$ such that $E^Q[X] = 0$. Then for any $\nu \in \text{dom}_+ U$ the maximizer in

$$\sup_{W \in \mathbb{R}^n} E[U(\nu + WX)]$$

exists.
Dependence of optimal investment on risk aversion

- We expect the amount of risky investment to fall with increasing aversion to risk.
- But at what rate?
- We can examine this dependence numerically by plotting the optimal investment $\alpha$ as a function of relative risk tolerance $1/R(v) = 1/\gamma$. 

![Graph showing the dependence of optimal investment on risk aversion](image)
Normalized portfolio and investment potential

- Similarly we can examine the dependence of the certainty equivalent on the risk aversion.
- This dependence again turns out to be close to linear.

**Definition**

1. For a given utility $U$, reference level $v$ and risky asset with excess return $X$ we define **normalized optimal portfolio** $\beta$ as the optimal risky investment $\hat{\alpha}$ per unit of local relative risk tolerance at the reference wealth:

   $$\beta := A(v) \hat{W} = R(v) \hat{\alpha}.$$

2. We define a normalized certainty equivalent gain, which we call the **investment potential**, as the percentage increase in certainty equivalent wealth per unit of risk tolerance,

   $$\text{IP} := A(v)(\text{CE}(\hat{\alpha}) - v) = R(v) \frac{\text{CE}(\hat{\alpha}) - v}{v}.$$
**Definition**

Consider \( v \) such that \( U'(v) > 0 \) and \( U''(v) < 0 \). We say that \( f \) given by the formula

\[
f(z) := c_1 U \left( v + \frac{z}{A(v)} \right) + c_2
\]

(3)

with

\[
c_1 := \frac{A(v)}{U'(v)}, \quad c_2 := -c_1 U(v),
\]

(4)

is a **normalized utility** to \( U \) at \( v \).

- The normalized utility \( f \) maps risk-free wealth \( v \) to 0 in such a way that we achieve unit risk aversion at 0,

\[
- \frac{f''(0)}{f'(0)} = 1.
\]
This is true regardless of the value $c_1$ and $c_2$. We pick $c_1$ and $c_2$ conveniently to obtain $f(0) = 0$ and $f'(0) = 1$.

It transpires that the normalized quantities can be computed by means of a normalized utility which we define next.

**Proposition (Brooks et al. 2006)**

Consider a utility function $U$ and the corresponding normalized utility $f$. In the absence of arbitrage

$$\hat{\beta}(X) = \arg \max_{\beta \in \mathbb{R}^n} \mathbb{E}[f(\beta X)],$$

$$\text{IP}(X) = f^{-1}(\mathbb{E}[f(\hat{\beta} X)]).$$
Normalized HARA utility

Proposition

The normalized utility is independent of \( \bar{V} \) and \( v \) and it is given by

\[
  f_\gamma(z) := \begin{cases} 
  \frac{(1+z/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma > 0, \gamma \neq 1, \\
  \ln(1+z) & \text{for } \gamma = 1, \\
  \frac{|1+z/\gamma|^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma < 0.
  \end{cases}
\]

The function \( f_\gamma(z) \) has a pointwise limit

\[
  f_\infty(z) := \lim_{|\gamma| \to \infty} f_\gamma(z) = 1 - e^{-z}, \text{ which is the normalized utility of } \text{CARA}_a \text{ for any } a > 0 \text{ and any } v \in \mathbb{R}.
\]

- Consequence: (normalized) optimal investment from CRRA\(\gamma\) is very similar to optimal investment from CARA when \(|\gamma|\) is large
- The same is true for the investment potential
Numerical example revisited

- In Matlab, enter:
  ```
  gama = 5;
  X = [0.18 -0.12];
  Xdistr = [0.5 0.5];
  ```

- Run the command:
  ```
  [IP beta] = HARAmx(X, XDistr, gama);
  ```

- This produces $\hat{\beta}(X) = 1.362$, $IP(X) = 0.020253$.

- To recover optimal investment and certainty equivalent for CRRA utility with $\gamma = 5$ we convert:

  \[
  \hat{\alpha} = \frac{\hat{\beta}}{R(v)} = \frac{1.362}{5} = 0.2724,
  \]

  \[
  CE = (1 + IP/R(v)) v
  = (1 + 0.020253/5) \times 1,220,000 = 1,224,942.
  \]
Consider our numerical example

- Compute IP and $\hat{\beta}$ for different values of $\gamma$
- Investment potential is a robust measure

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0.5</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>15</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\beta}_\gamma$</td>
<td>1.389</td>
<td>1.389</td>
<td>1.375</td>
<td>1.362</td>
<td>1.355</td>
<td>1.352</td>
</tr>
<tr>
<td>IP$_\gamma$</td>
<td>0.0208</td>
<td>0.0206</td>
<td>0.0204</td>
<td>0.0203</td>
<td>0.0202</td>
<td>0.0201</td>
</tr>
</tbody>
</table>
Quadratic utility I

- Special case of HARA utility with $\gamma = -1$

  \[ \text{HARA}_{-1, \bar{V}}(V) = -\frac{(\bar{V} - V)^2}{2} \]

- It is the only utility that does not require numerical solutions when the market is incomplete

- Quadratic utility has a **bliss point** at $\bar{V}$

- Local relative risk aversion

  \[ R(v) = (\bar{V}/v - 1)^{-1} \]

- Investment potential

  \[ \text{IP}_{-1}(X) = \max_{\beta} 1 - \sqrt{\mathbb{E}[(1 - \beta X)^2]} \]
Quadratic utility II

- Normalized optimal investment
  \[ \hat{\beta}_{-1} = \frac{E[X]}{E[X^2]} . \]

- Numerically, in our example
  \[
  \hat{\beta}_{-1}(X) = \frac{E[X]}{E[X^2]} = \frac{0.5 \times (0.18 - 0.12)}{0.5 \times (0.18^2 + 0.12^2)} = 1.282,
  \]

- The investment potential generated by quadratic utility is
  \[
  \text{IP}_{-1}(X) = 1 - \sqrt{1 - \frac{(E[X])^2}{E[X^2]}} = 0.0194
  \]
Investment potential and Sharpe ratio

- A simple manipulation yields \(1 - \frac{(E[X])^2}{E[X^2]} = \frac{1}{1 + SR^2(X)}\)
- Consequently

\[
IP_{-1}(X) = 1 - \sqrt{\frac{1}{1 + SR^2(X)}}
\]

- This works specifically for quadratic utility
- We can also try asymptotic analysis for small Sharpe ratio

\[
\frac{CE - \nu}{\nu} \approx \frac{1}{2} \frac{SR^2(X)}{R(\nu)}
\]

- The asymptotics work for any utility function
Problems with Sharpe ratio

- Because of the bliss point on the quadratic utility Sharpe ratio may underestimate true investment potential.
- This will happen when the wealth of optimal portfolio reaches beyond the bliss point:
  \[ 1 - \beta_{-1} X < 0 \]
- Depending on the sign of \( \beta_{-1} \) this will happen when:
  \[ X_{\text{max}} > \frac{1}{\beta_{-1}} \quad \text{for} \quad \beta_{-1} > 0 \]
  \[ X_{\text{min}} < \frac{1}{\beta_{-1}} \quad \text{for} \quad \beta_{-1} < 0 \]
- In such case one can increase the SR by throwing money away in good states.
Arbitrage-adjusted Sharpe ratio

- Consider excess returns of two assets, A and B

<table>
<thead>
<tr>
<th>Probability</th>
<th>1/6</th>
<th>1/2</th>
<th>1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess return of Asset A</td>
<td>-1%</td>
<td>1%</td>
<td>2%</td>
</tr>
<tr>
<td>Excess return of Asset B</td>
<td>-1%</td>
<td>1%</td>
<td>11%</td>
</tr>
</tbody>
</table>

- We find $\text{SR}_A > \text{SR}_B$
- However, asset B stochastically dominates asset A!
- The solution is to separate the excess return into 2 parts
  - Part with maximum Sharpe ratio
  - Pure arbitrage excess return (wealth we have set aside)
- We keep disposing of wealth in good states until the bliss point condition is just met
- See book 3.6.2-3.6.5
