Expected Utility and Performance Measures

Aleš Černý

Cass Business School
CITY UNIVERSITY LONDON

Energy & Finance Seminar, Essen
25th November 2009

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  - Certainty equivalent growth rate
  - Investment potential
  - Sharpe ratio and its shortcomings

Reading

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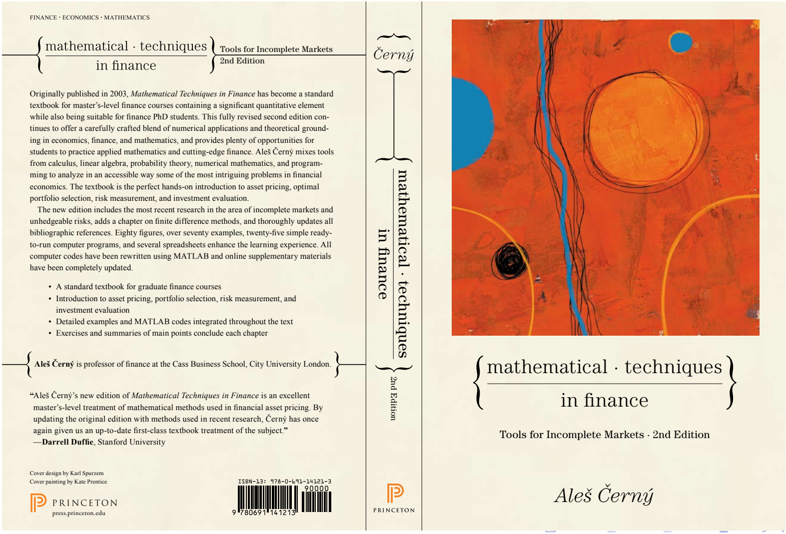
- Textbook, chapter 3: Černý 2009



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- Detailed examples and MATLAB codes integrated throughout the text
- Exercises and summaries of main points conclude each chapter

Aleš Černý is professor of finance at the Cass Business School, City University London.

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—**Darrell Duffie**, Stanford University

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press.princeton.edu

ISBN-13: 978-0-691-14121-3

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An investor is risk averse when positive deviations from her average wealth do not compensate for equally large and equally probable negative deviations

- The two assumptions are captured by a concave and increasing function U, commonly called a utility function

Utility function

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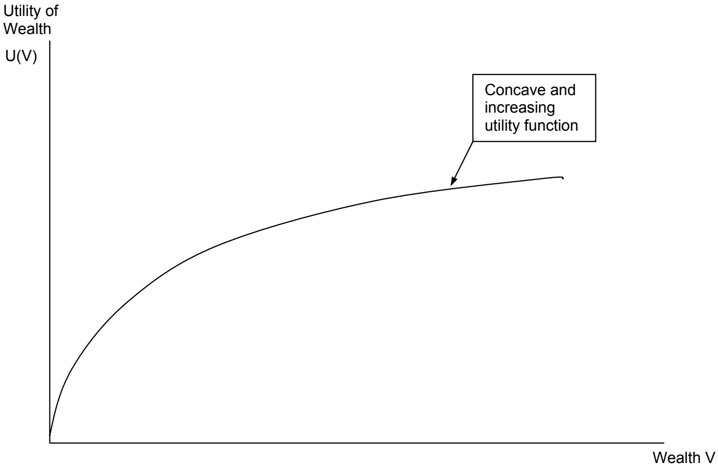

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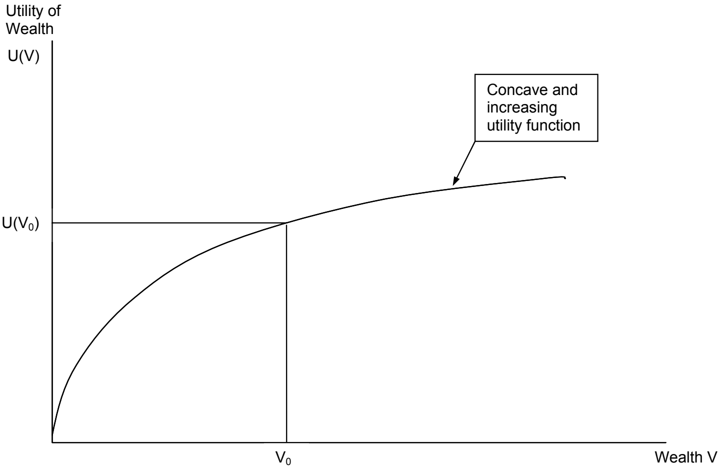

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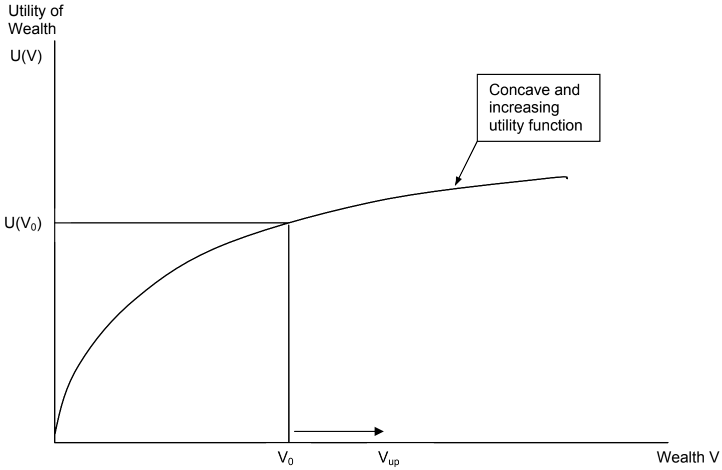

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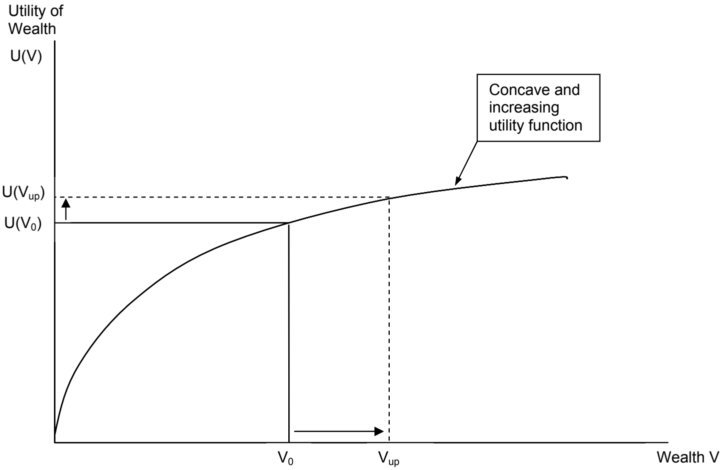

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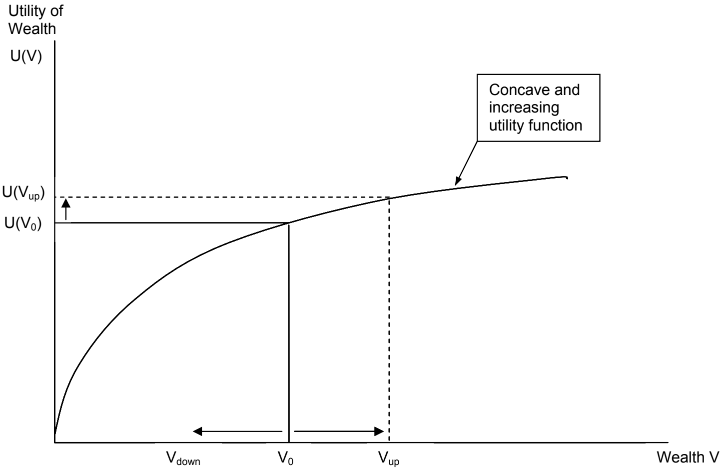

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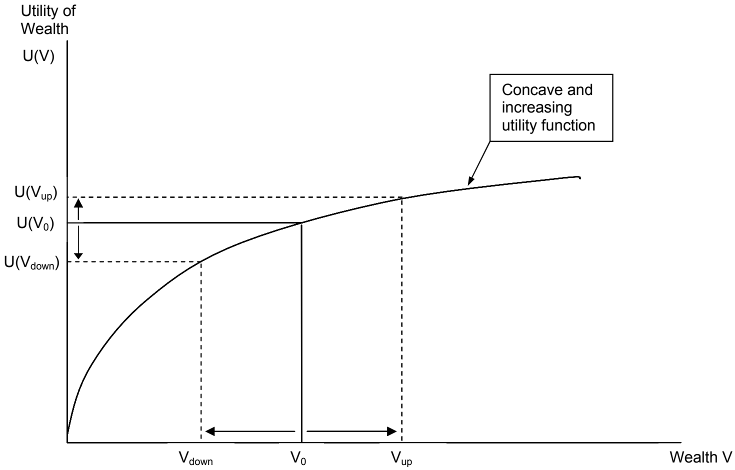

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$$\begin{aligned}
\mathrm{HARA}_{\gamma,\bar{V}}(V) &= \frac{(\bar{V}+V)^{1-\gamma}}{1-\gamma} \text{with } \gamma > 0, \\
\mathrm{HARA}_{\gamma,\bar{V}}(V) &= \frac{|\bar{V}-V|^{1-\gamma}}{1-\gamma} \text{with } \gamma < 0
\end{aligned}$$

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$$U(\mathrm{CE}) = \mathrm{E}[U(V)]$$

Expected utility and certainty equivalent wealth

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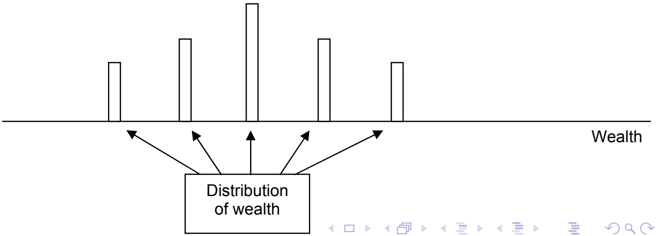

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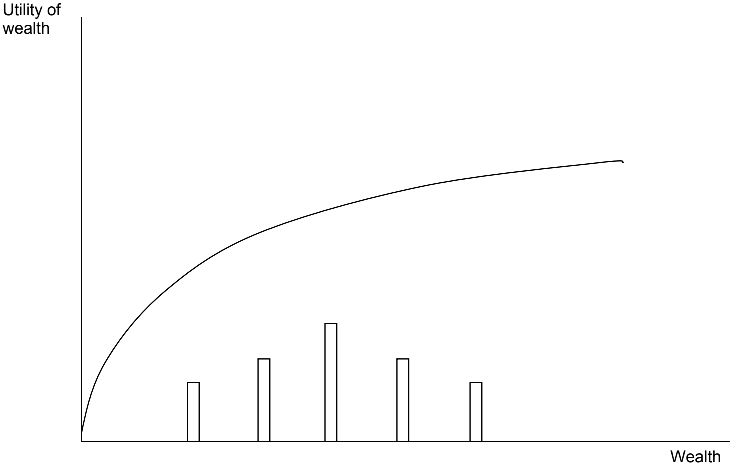

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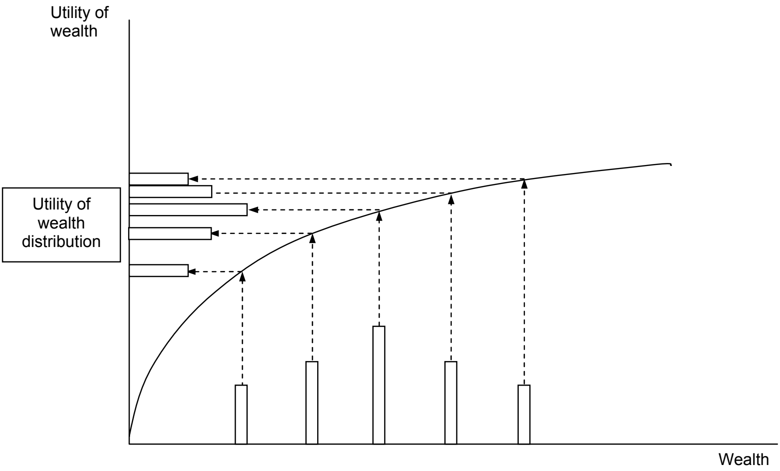

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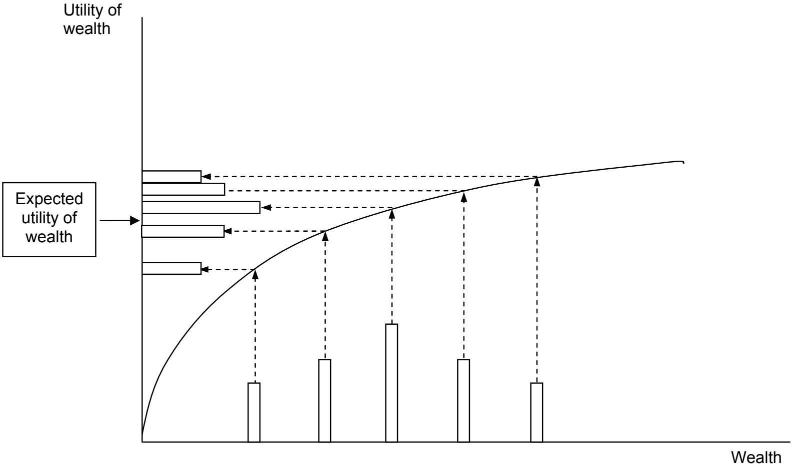

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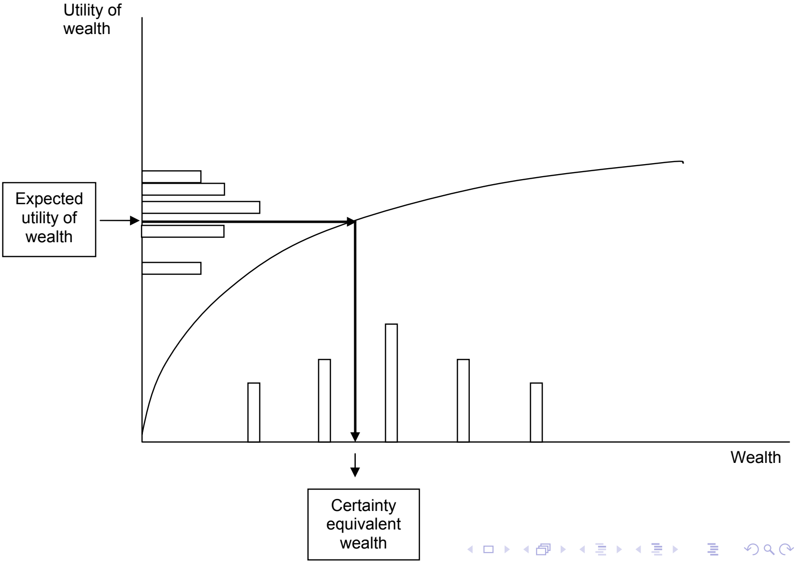

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$$\mathrm{CE} - v = \frac{1}{2}\frac{U''(v)}{U'(v)}\sigma^2 + o(\sigma^2)$$

- We call $A(v) := -\frac{U''(v)}{U'(v)}$ the coefficient of local absolute risk aversion

Measurement of risk aversion II

- Derivation: write down 2nd order Taylor expansion

$$U(v+\epsilon) = U(v) + U'(v)\epsilon + \frac{1}{2}U''(v)\epsilon^2 + o(\epsilon^2)$$

- Take expectations on both sides

$$\mathrm{E}[U(v+\epsilon)] = U(v) + \frac{1}{2}U''(v)\sigma^2 + o(\sigma^2)$$

- Write down 1st order expansion for the certainty equivalent

$$U(\mathrm{CE}) = U(v+\mathrm{CE}-v) = U(v) + U'(v)(\mathrm{CE}-v) + o(\mathrm{CE}-v)$$

- From $U(\mathrm{CE}) = \mathrm{E}[U(v+\epsilon)]$ we find

$$\mathrm{CE} - v = \frac{1}{2}\frac{U''(v)}{U'(v)}\sigma^2 + o(\sigma^2).$$

- The difference $\mathrm{CE} - v$ is the risk premium

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- Optimal portfolio selection is about balancing risk and reward
- Mathematically this is achieved by maximizing expected utility of terminal wealth

Expected utility maximization - numerical example

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- You wish to invest your savings for a year
- You can invest either in safe account with rate of return 2% p.a.
- Or a risky stock, returning either 20% or -10% with equal probability
- Your task:
  1. Write down how much (out of your £1 million) you would invest in the stock
  2. Calculate how much a person with utility function $U(V) = -V^{-4}/4$ should invest in the stock

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  - function U with this property is called closed or upper semi-continuous
- Example of a discontinuous but closed concave function

$$U(x) = \begin{cases} \sqrt{x} & \text{for} \quad x \geq 0, \\ -\infty & \text{for} \quad x < 0, \end{cases}$$

Existence of optimal portfolios II

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- Concave functions need not be differentiable at every interior point x of dom U but they always possess left and right derivatives

$$
\begin{aligned}
U'_+(x) \; &:= \lim_{h\to 0_+} \frac{U(x+h)-U(x)}{h}, \\
U'_-(x) \; &:= \lim_{h\to 0_-} \frac{U(x+h)-U(x)}{h}.
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  - Outside the effective domain we set:

$$
\begin{aligned}
U'_-(x) &= U'_+(x) = \infty \text{ for } x < \inf \operatorname{dom} U, \\
U'_-(x) &= U'_+(x) = -\infty \text{ for } x > \sup \operatorname{dom} U.
\end{aligned}
$$

Existence of optimal portfolios III

Theorem (Černý et al. 2008)

Suppose $U : \mathbb{R} \to [-\infty, \infty)$ *is a closed concave function and there is an open interval* $\mathrm{dom}_+ U$ *on which* U *is strictly increasing. Assume*

$$\frac{U'_+(\infty)}{U'_-(-\infty)} \leq 0,$$

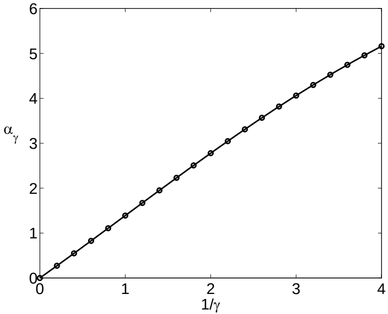
where we adopt the convention $\frac{-\infty}{\infty} \leq 0$.
Let X *be an* $\mathbb{R}^n$*-valued bounded random variable and suppose there exists a probability measure* Q *such that* $E^Q[X] = 0$. *Then for any* $v \in \mathrm{dom}_+ U$ *the maximizer in*

$$\sup_{W \in \mathbb{R}^n} E[U(v + WX)]$$

exists.

Dependence of optimal investment on risk aversion

- We expect the amount of risky investment fall with increasing aversion to risk
- But at what rate?
- We can examine this dependence numerically by plotting the optimal investment α as a function of relative risk tolerance $1/R(v) = 1/\gamma$.



Normalized portfolio and investment potential

- Similarly we can examine the dependence of the certainty equivalent on the risk aversion.
- This dependence again turns out to be close to linear.

Definition

1. For a given utility U, reference level v and risky asset with excess return X we define **normalized optimal portfolio** $\hat{\beta}$ as the optimal risky investment $\hat{\alpha}$ per unit of local relative risk tolerance at the reference wealth:

$$\hat{\beta} := A(v)\hat{W} = R(v)\hat{\alpha}. \tag{1}$$

2. We define a normalized certainty equivalent gain, which we call the **investment potential**, as the percentage increase in certainty equivalent wealth per unit of risk tolerance,

$$\text{IP} := A(v)(\text{CE}(\hat{\alpha}) - v) = R(v)\frac{\text{CE}(\hat{\alpha}) - v}{v}. \tag{2}$$

Normalized utility I

Definition

Consider v such that $U'(v) > 0$ and $U''(v) < 0$. We say that f given by the formula

$$f(z) := c_1 U\left(v + z/A(v)\right) + c_2 \tag{3}$$

with

$$c_1 := \frac{A(v)}{U'(v)}, c_2 := -c_1 U(v), \tag{4}$$

is a **normalized utility** to U at v.

- The normalized utility f maps risk-free wealth v to 0 in such a way that we achieve unit risk aversion at 0,

$$-\frac{f''(0)}{f'(0)} = 1.$$

Normalized utility II

- This is true regardless of the value c_1 and c_2. We pick c_1 and c_2 conveniently to obtain $f(0) = 0$ and $f'(0) = 1$.
- It transpires that the normalized quantities can be computed by means of a normalized utility which we define next.

Proposition (Brooks et al. 2006)

Consider a utility function U and the corresponding normalized utility f. In the absence of arbitrage

$$\begin{aligned} \hat{\beta}(X) &= \arg\max_{\beta \in \mathbb{R}^n} \mathrm{E}[f(\beta X)], \\ \mathrm{IP}(X) &= f^{-1}(\mathrm{E}[f(\hat{\beta} X)]). \end{aligned}$$

Normalized HARA utility

Proposition

The normalized utility is independent of $\bar{V}$ and v and it is given by

$$f_\gamma(z) := \begin{cases} \frac{(1+z/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma > 0, \gamma \neq 1, \\ \ln(1+z) & \text{for } \gamma = 1, \\ \frac{|1+z/\gamma|^{1-\gamma}-1}{1/\gamma-1} & \text{for } \gamma < 0. \end{cases}$$

The function $f_\gamma(z)$ has a pointwise limit $f_\infty(z) := \lim_{|\gamma|\to\infty} f_\gamma(z) = 1 - e^{-z}$, which is the normalized utility of CARA_a for any $a > 0$ and any $v \in \mathbb{R}$.

- Consequence: (normalized) optimal investment from CRRA_γ is very similar to optimal investment from CARA when $|\gamma|$ is large
- The same is true for the investment potential

Numerical example revisited

- In Matlab, enter

```
gama   = 5;
X      = [0.18 -0.12];
Xdistr = [0.5 0.5];
```

- Run the command

```
[IP beta] = HARAmax(X,XDistr,gama);
```

- This produces $\hat{\beta}(X) = 1.362, \qquad \mathrm{IP}(X) = 0.020253.$
- To recover optimal investment and certainty equivalent for CRRA utility with $\gamma = 5$ we convert

$$
\begin{aligned}
\hat{\alpha} &= \frac{\hat{\beta}}{R(v)} = \frac{1.362}{5} = 0.2724, \\
\mathrm{CE} &= (1 + \mathrm{IP}/R(v))\, v \\
&= (1 + 0.020253/5) \times 1,220,000 = 1,224,942.
\end{aligned}
$$

Performance measurement in one-period models II

- Consider our numerical example
- Compute IP and $\hat{\beta}$ for different values of γ
- Investment potential is a robust measure

γ	**0.5**	**1**	**2**	**5**	**15**	∞
$\hat{\beta}_\gamma$	1.389	1.389	1.375	1.362	1.355	1.352
IP_γ	0.0208	0.0206	0.0204	0.0203	0.0202	0.0201

Quadratic utility I

- Special case of HARA utility with $\gamma = -1$

$$\mathrm{HARA}_{-1,\bar{V}}(V) = -\frac{(\bar{V} - V)^2}{2}.$$

- It is the only utility that does not require numerical solutions when the market is incomplete
- Quadratic utility has a bliss point at $\bar{V}$
- Local relative risk aversion

$$R(v) = (\bar{V}/v - 1)^{-1}$$

- Investment potential

$$\mathrm{IP}_{-1}(X) = \max_{\beta} 1 - \sqrt{\mathrm{E}[(1 - \beta X)^2]}$$

Quadratic utility II

- Normalized optimal investment

$$\hat{\beta}_{-1} = \frac{\mathrm{E}[X]}{\mathrm{E}[X^2]}.$$

- Numerically, in our example

$$\hat{\beta}_{-1}(X) = \frac{\mathrm{E}[X]}{\mathrm{E}[X^2]} = \frac{0.5\,(0.18 - 0.12)}{0.5\,(0.18^2 + 0.12^2)} = 1.282,$$

- The investment potential generated by quadratic utility is

$$\mathrm{IP}_{-1}(X) = 1 - \sqrt{1 - \frac{(\mathrm{E}[X])^2}{\mathrm{E}[X^2]}} = 0.0194$$

Investment potential and Sharpe ratio

- A simple manipulation yields $1 - \frac{(\mathrm{E}[X])^2}{\mathrm{E}[X^2]} = 1/(1 + \mathrm{SR}^2(X))$
- Consequently

$$\mathrm{IP}_{-1}(X) = 1 - \sqrt{1/(1 + \mathrm{SR}^2(X))}$$

- This works specifically for quadratic utility
- We can also try asymptotic analysis for small Sharpe ratio

$$\frac{\mathrm{CE} - v}{v} \approx \frac{1}{2}\mathrm{SR}^2(X)/R(v)$$

- The asymptotics work for any utility function

Problems with Sharpe ratio

- Because of the bliss point on the quadratic utility Sharpe ratio may underestimate true investment potential
- This will happen when the the wealth of optimal portfolio reaches beyond the bliss point

$$1 - \hat{\beta}_{-1} X < 0$$

- Depending on the sign of $\hat{\beta}_{-1}$ this will happen when

$$\begin{aligned} X_{\max} &> 1/\hat{\beta}_{-1} \quad \text{for } \hat{\beta}_{-1} > 0 \\ X_{\min} &< 1/\hat{\beta}_{-1} \quad \text{for } \hat{\beta}_{-1} < 0 \end{aligned}$$

- In such case one can increase the SR by throwing money away in good states

Arbitrage-adjusted Sharpe ratio

- Consider excess returns of two assets, A and B

Probability	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
Excess return of Asset A	-1%	1%	2%
Excess return of Asset B	-1%	1%	11%

- We find $SR_A > SR_B$
- However, asset B stochastically dominates asset A!
- The solution is to separate the excess return into 2 parts
  - Part with maximum Sharpe ratio
  - Pure arbitrage excess return (wealth we have set aside)
- We keep disposing of wealth in good states until the bliss point condition is just met
- See book 3.6.2-3.6.5

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