Computation of Optimal Monotone Mean-Variance Portfolios via Truncated Quadratic Utility

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Energy & Finance Seminar, Essen
25th November 2009

Goals of the talk

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- Understand more advanced aspects of portfolio selection

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  - mean-variance utility and its monotonization

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- Find out about key mathematical concepts of convex optimization

Reading

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- The "Rock" Rockafellar (1996)

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- The paper Černý et al. (2008)

Introduction I

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- The said non-monotonicity is a major drawback of these classical utility functions
- The first approach, Černý (2003), uses expected truncated quadratic utility and leads to the so-called arbitrage-adjusted Sharpe ratio.
- The second, formulated in Maccheroni et al. (2009), modifies the variational form of mean-variance preferences

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- In this paper we show that there is an important and useful link between the optimal portfolios
- This link is all the more interesting because variational preferences are closely related to convex risk measures (see Föllmer and Schied 2002, Föllmer et al. 2009, and Frittelli and Rosazza Gianin 2002)

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- Mean-variance utility is

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- The two are obviously related

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$$f^*(z) = \inf_{x \in \mathbb{R}} \{xz - f(x)\}$$

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- In order to study the preference functional Φ we will restrict our attention to the following class of functions.

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$\mathcal{U}$ denotes the set of functions f satisfying (A1) and such that $f(0) = 0$, $f'_{+}(0) \leq 1 \leq f'_{-}(0)$, and there exist $x < 0 < y$ in dom f with $f'_{+}(x) > 1$ and $1 > f'_{+}(y) > 0$.

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- For example, $f \in \mathcal{U}$ if it is twice continuously differentiable around 0, with $f''(0) < f(0) = 0$ and $f'(0) = 1$.
- $f \in \mathcal{U}$ implies $1 \in \operatorname{int}\operatorname{dom} f^*$ and f^* attains its supremum at 1, with $f^*(1) = 0$

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Theorem

If $f \in \mathcal{U}$, then

$$\Phi(Y) = \max_{\eta \in [\operatorname{essinf} Y, \operatorname{ess\,sup} Y]} \{\eta + F(Y - \eta)\}, \quad \forall Y \in L^{\infty}(P).$$

Moreover, Φ is concave, increasing, translation invariant, finite, and Lipschitz.

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- Our aim is to determine the relations between the maximizers and optimal values

$$\hat{\alpha}_X \in \arg\max_{\alpha \in \mathbb{R}^n} F_X(\alpha) \qquad \text{and} \qquad \hat{F}_X = F_X(\hat{\alpha}_X),$$

$$\hat{\beta}_X \in \arg\max_{\beta \in \mathbb{R}^n} \Phi_X(\beta) \qquad \text{and} \qquad \hat{\Phi}_X = \Phi_X(\hat{\beta}_X).$$

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Suppose that $X \in L^\infty(P)^n$ *is arbitrage free and that* $f'(\infty)/f'(-\infty) = 0$. *Then,* $\arg\max_{\alpha \in \mathbb{R}^n} F_X(\alpha) \neq \emptyset$. *Moreover,* $f(0) \leq \hat{F}_X < f(\mathrm{sd}_+ f)$.

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Suppose that $X \in L^\infty(P)^n$ is arbitrage free and that $f'(\infty)/f'(-\infty) = 0$. Then, $\arg\max_{\alpha \in \mathbb{R}^n} F_X(\alpha) \neq \emptyset$. Moreover, $f(0) \leq \hat{F}_X < f(\mathrm{sd}_+ f)$.

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Suppose that $X \in L^\infty(P)^n$ is arbitrage free. If f belongs to $\mathcal{U}$, with $f'(\infty) = 0$ and $f'(-\infty) = \infty$, then $\arg\max_{\beta \in \mathbb{R}^n} \Phi_X(\beta) \neq \emptyset$.

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- The above implies the existence of $\hat{\beta}_X$ and $\hat{\eta}_X$ such that

$$\hat{\Phi}_X = \Phi(\hat{\beta}_X X) = \max_{\eta \in \mathbb{R}, \beta \in \mathbb{R}^n} \{\eta + F(\beta X - \eta)\} = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X).$$

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- The quantity $\hat{\eta}_X$ may in general depend on $\hat{\beta}_X$ if the latter is not unique

Portfolio selection IV

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Suppose that $f \in \mathcal{U}$ and that $X \in L^\infty(P)^n$ is arbitrage free. If $\hat{\beta}_X$ and $\hat{\eta}_X$ satisfy

$$\hat{\Phi}_X = \hat{\eta}_X + F(\hat{\beta}_X X - \hat{\eta}_X),$$

then $-\hat{\eta}_X \in \mathrm{dom}_+ f$.

- At this point we need to impose a specific structure on f to progress further

Main result

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- Consider normalized truncated HARA utility

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f_\gamma(x) = \begin{cases} \frac{(1+x/\gamma)^{1-\gamma}-1}{1/\gamma-1} & \text{for } x < -\gamma \\ \frac{\gamma}{\gamma-1} & \text{for } x > -\gamma \end{cases}, \qquad \gamma < 0, \text{ and}
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(ii) *the optimal values $\hat{F}_{\gamma,X}$ and $\hat{\Phi}_{\gamma,X}$ are one-to-one:*

$$\hat{\Phi}_{\gamma,X} = \begin{cases} \frac{\gamma^2}{1-\gamma}\left(\left(\hat{F}_{\gamma,X}\,(1/\gamma - 1) + 1\right)^{1/\gamma} - 1\right) & \text{for } \gamma \in \mathbb{R}\setminus\{0,1\} \\ \hat{F}_{\gamma,X} & \text{for } \gamma = 1 \\ -\ln(1 - \hat{F}_{\gamma,X}) & \text{for } \gamma = \pm\infty \end{cases}$$

Main result IV

Theorem (continued)

(iii) *the optimal portfolios for the two criteria are related as follows*

$$\hat{\beta}_{\gamma,X} = \begin{cases} \hat{\alpha}_{\gamma,X}\left(\hat{F}_{\gamma,X}\left(1/\gamma - 1\right) + 1\right)^{1/\gamma} & \text{for } \gamma \in \mathbb{R}\backslash\{0,1\} \\ \hat{\alpha}_{\gamma,X} & \text{for } \gamma = 1 \\ \hat{\alpha}_{\gamma,X} & \text{for } \gamma = \pm\infty \end{cases}$$

where the equality is to be interpreted as equality of sets in $\mathbb{R}^n$.

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  - Remarkably, $\hat{\alpha}_{\gamma,X}$ and $\hat{\beta}_{\gamma,X}$ feature the same mix of risky assets, though the leverage is different

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$$\widehat{\Phi}_{-1,X} = \frac{1}{2}\mathrm{SR}^2_{\mathrm{m}}(\hat{\alpha}_{-1,X}X),$$
$$\hat{\beta}_{-1,X} = \hat{\alpha}_{-1,X}(1 + \mathrm{SR}^2_{\mathrm{m}}(\hat{\alpha}_{-1,X}X))^{-1}.$$

- The $n+1$ equations which characterize the optimal value $\hat{\beta}_{-1,X}$ in Maccheroni et al. (2009) are now readily seen to be the first order conditions of the optimization over η and β

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