Examples for applying Lévy processes to financial problems

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Motivation

Many structured products and financial innovation for private, institutional investors but also in the interbanking market.

Markets exhibit smiles, skews, fat tails and other features which suggest that we need non-normal modeling.

Different markets (Equity, FX, Interest Rate, Credit or Hybrid) need different modeling.

High quality risk management and advanced financial models are necessary!

Motivation

Market consistent valuation and risk management is very demanding and a setup must include:

- Market Data
- Model Choice
- Calibration
- Pricing
- Hedging / Risk Management

Aim of the talk

We consider each stage for CPPI based selection and pricing complex equity linked structures in non-normal markets

Outline of the Talk

- CPPI and (Exotic) Equity Options
- Market Data -
  - Available Market Data
  - The Volatility Surface
  - Appropriate Usage of Market Data
- Models
  - Diffusion Models
  - Jump-Diffusion Models
  - Stochastic Volatility Models
  - Lévy Models
  - Stochastic Volatility Lévy Models
- Pricing
- Calibration
- Hedging

CPPI Mechanism

CPPI is the abbrevation for *Constant Proportion Portfolio Insurance* The CPPI mechanism is a rules-based trading strategy. It seeks to maximise returns by way of leveraged exposure to a (portfolio) of risky asset(s) and providing a principal protection. This takes place in certain risk thresholds. The risks are known as gap risk.

The are many modifications of the basic CPPI rules around!

CPPI Vocabulary

Bond Floor
The value of a Zerobond with the same time to maturity as the CPPI strategy. Could also be a coupon bearing bond.

Cushion
The cushion is the difference of the Bond Floor and the current value of the CPPI insured portfolio

Leverage Factor (Multiplier)
The LF is the factor multiplied with the cushion to give the possible amount to be invested in the risky assets. It represents the overnight risk inherent in the risky assets. Protection Level. This is the amount of principal which should be protected. In classical CPPI the PL = 100

CPPI Vocabulary

Maximum Exposure
The ME of the CPPI is the maximum level to which the capital is invested into the risky assets

Minimum Exposure The ME of the CPPI is the minimum level to which the capital is invested into the risky asset. For classical CPPI the ME $= 0$.

Lock In The Lock-In mechanism allows to lock in an upside already achieved during the lifetime of the CPPI

Deleverage
Deleverage is the event occuring if Cushion $= 0$. Then the portfolio is only worth the BF

Coupons in CPPI

- To achieve periodic payments the basic CPPI strategy could be modified to pay (half-) yearly coupons linked to LIBOR, e.g. LIBOR + 50 bp.
- Increases the risk of deleveraging, since one takes money out which decreases the cushion periodically!
- The coupon is not guaranteed, e.g. would only be paid if the strategy would not deleverage.
- A new risk arises, namely coupon shortfall

In our CPPI setting we examine Deleverage Probability, Coupon Shortfall Probability and Return

The Plan

For real applications we consider a basket of risky assets. This allows to co-movement to increase the overall return and to reduce deleverage and coupon shortfall probability.

- Determine the universe of risky assets (mainly qualitative)
- Analysis of the universe using methods from time series analysis (mean, volatility, skew, kurtosis, correlation, etc.)
- Asset Allocation Approach to determine the efficient frontier
- Simulate the CPPI Mechanism for „optimal" portfolios

The Simplest Setting

- Assume a Gaussian world and determine the mean vector and the covariance matrix
- Compute the Markowitz efficient frontier
- Run a one-factor simulation along the efficient frontier using Mean basket = sum basket const Variance basket = sum cov(basket const, basket const)

In this setting we do not allow skewed or fat tailed distributions!

Options I - Cliquet Options

For a given asset $S(t)$, $t \in [0, T]$ we consider the performance on the time interval $[t, t + \Delta]$: $P(t) := \frac{S(t+\Delta)}{S(t)} - 1$
Fix observation points $t_k \in [0, T)$, $k = 1, \ldots, N$ and consider $P(t_k)$. A locally capped/floored (lc/lf) Cliquet option has payoff:

$$R := \sum_{k=1}^{N-1} \max(\min(P(t_k), lc), lf)$$

A globally capped/floored (gc/gf) Cliquet option has payoff:

$$\max(\min(R, gc), gf)$$

There are several structures derived from the simple Cliquet option such as Swing Cliquet, Reverse Cliquet.

Options II - Digital Options

Let $K > 0$, the strike, be a non-negative number. The payoff of a Digital option is:

$$h(S(T)) = 1_{\{S(T)<K\}}$$

Digitals are exposed to the volatility skew / smile since

$$h(S(T)) \approx \frac{1}{\epsilon}\left(\mathrm{Call}(K-\epsilon) - \mathrm{Call}(K)\right)$$

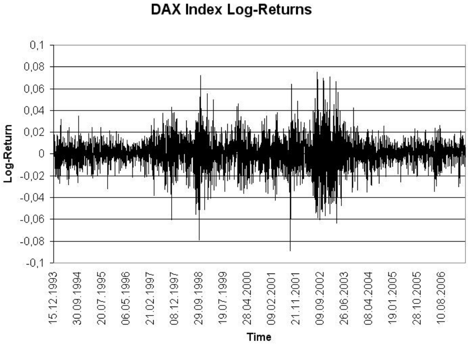
Options III -Barrier Options

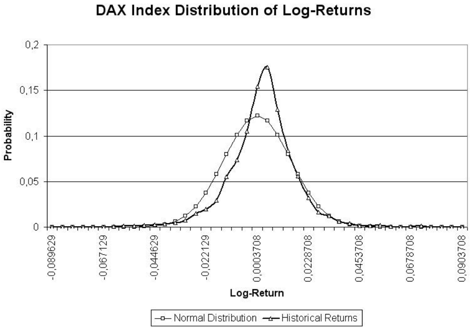
Let $M(t) := \sup_{u\in[0,t]} S(u)$, $m(t) := \inf_{u\in[0,t]} S(u)$ and $K > 0$.
The payoff of a Knock-Out call is:

$$h(S(T)) = (S(T) - K)^{+} 1_{\{M(t)<U(t) \text{ for all } t\in[0,T]\}}$$
$$h(S(T)) = (S(T) - K)^{+} 1_{\{L(t)<m(t) \text{ for all } t\in[0,T]\}}$$

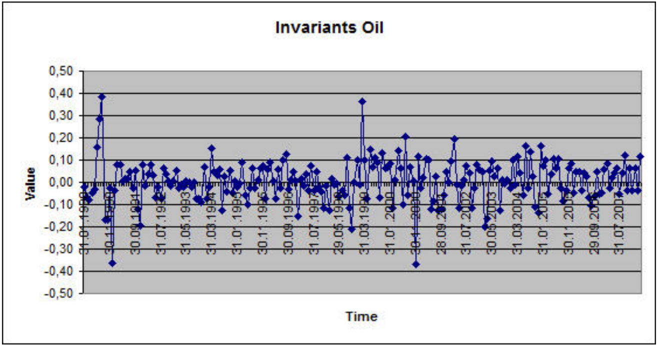
Knock-Out puts, Knock-In options and more complex Barrier options such as Corridor options or options with several knock-out / knock-in features can be considered.

Market Data I - Time Series

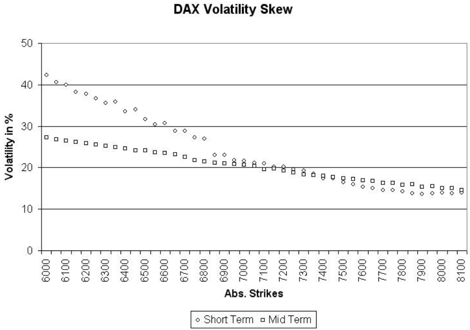


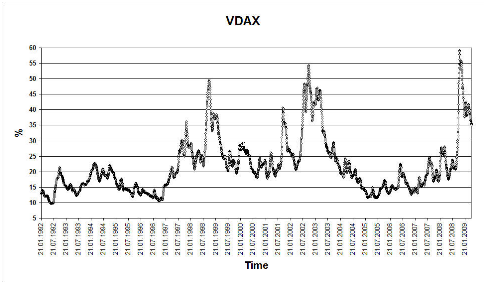


Market Data I - Time Series

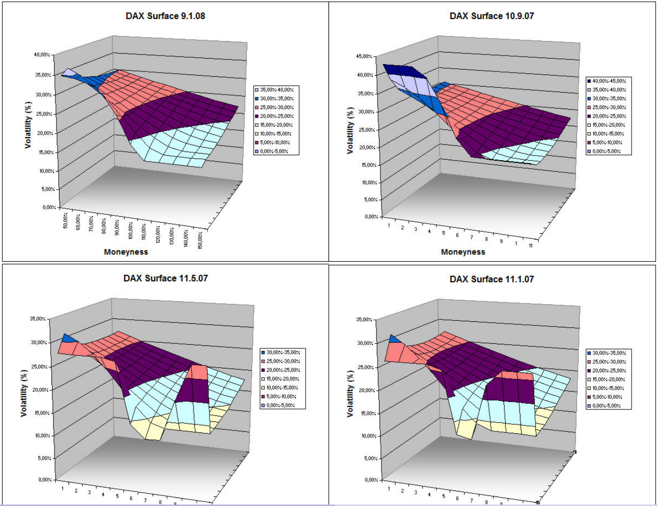


Market Data II - Option Data





Market Data III - Volatility Skew



Using Market Data

Which market data to choose?

- Historical Data (time series)
- Quoted Option Prices
- Liquid Options / Illiquid Options
- Broker Prices for Exotics

Basic Modeling Setup

We consider equity models with deterministic interest rates. Let r be the riskless rate and d the dividend yield. We consider the stochastic process $S(t)_t$ or the log process $X(t)_t$ given by:

$$\begin{aligned} S(t) &= S(0)\exp((r-d)t + L(t)) \\ X(t) &= X(0) + (r-d)t + L(t) \end{aligned}$$

We give examples for choosing $L(t)_t$ by considering diffusion, jump-diffusion or pure jump processes.

Including stochastic interest rates (equity - interest rate hybrids) can also be considered, for example the Heston-Hull-White model (see e.g. Kammeyer and Kienitz (2009)).

Models

How to choose a model?

1. Theoretical features (Skews, Tails,...)
2. Numerical / Analytical Tractability
3. Fitting Option Prices
4. Hedge Strategies

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Models I - Basic Models

- Black-Scholes-Merton Model

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + \sigma S(t)dW(t) \\ S(0) &= S_0 \end{aligned}$$

- Merton Jump Diffusion Model

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + \sigma S(t)dW(t) + dJ(t) \\ S(0) &= S_0 \end{aligned}$$

Models II - Local Volatility Models

- CEV Models

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + \sigma S(t)^{\beta} dW(t) \\ S(0) &= S_0 \end{aligned}$$

- Displaced Diffusion Models

$$\begin{aligned} dS(t) &= (r-d)(S(t)+a)dt + \sigma(S(t)+a)dW(t) \\ S(0) &= S_0 \end{aligned}$$

Models III - Stochastic Volatility Models

- Heston Model

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + \sqrt{V(t)}S(t)dW_1(t) \\ dV(t) &= \kappa(\Theta - V(t))dt + \nu\sqrt{V(t)}dW_2(t) \\ S(0) &= S_0 \\ V(0) &= V_0 \end{aligned}$$

- Bates Model

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + \sqrt{V(t)}S(t)dW_1(t) + dJ(t) \\ dV(t) &= \kappa(\Theta - V(t))dt + \nu\sqrt{V(t)}dW_2(t) \\ S(0) &= S_0 \\ V(0) &= V_0 \end{aligned}$$

Models IV - Lévy Models

Lévy models are not specified by a Stochastic Differential Equation. They are represented by the characteristic function instead. Well known models include:

- Variance Gamma Model (VG); $\left(\frac{GM}{GM+(M-G)iu+u^2}\right)^C$
- Normal Inverse Gaussian Model (NIG); $\exp\left(-\delta\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta}\right)$
- Meixner Model (MX); $\frac{2\cos(\beta/2)}{\cosh((\alpha u-i\beta)/2)}$
- CGMY Model (CGMY); $\exp\left(C\Gamma(-Y)\left((M-iu)^Y-M^Y+(G+iu)^Y-G^Y\right)\right)$

Models V - Stochastic Volatility Lévy Models

To incorporate stochastic volatility into a model based on a Lévy process we let the time be a (independent) stochastic process which can be seen as modeling business time. It may also reflect the stochastic movement of the index's volatility index.

The model we consider is given by:

$$\begin{aligned} S(t) &= S(0)\exp((r-d)t + L(Y(t))) \\ X(t) &= X(0) + (r-d)t + L(Y(t)) \end{aligned}$$

Examples for stochastic clocks:

- Integrated Gamma Ornstein-Uhlenbeck process (GOU)
- Integrated Cox-Ingersoll-Ross process (CIR)

Stochastic Clock

- Gamma Ornstein-Uhlenbeck clock (GOU)

$$dz(t) = -\lambda z(t)dt + dZ(\lambda t), \lambda > 0, Z \text{ being compound Poisson}$$

$$\varphi_{GOU} = \exp\left(\frac{iuy_0}{\lambda}(1-\exp(-\lambda t)) + \frac{\lambda a}{iu-\lambda b}\left(b\log\left(\frac{b}{b-\frac{iu}{\lambda}(1-\exp(-\lambda t))}\right) - iut\right)\right)$$

- Cox-Ingersoll-Ross clock (CIR)

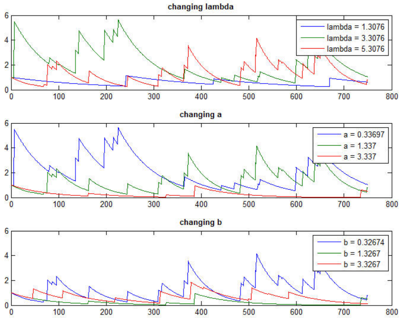
$$\varphi_{CIR} = \exp\left(\frac{iuy_0}{\lambda}(1-\exp(\lambda t)) + \frac{2aiu}{b\lambda}A(u,t)\right)$$

$$A(u,t) = \frac{1-\sqrt{1+\kappa}(1-\exp(-\lambda t))}{\kappa} + \frac{1}{\sqrt{1+\kappa}}\left(\operatorname{arctanh}\left(\frac{\sqrt{1+\kappa(1-\exp(-\lambda t))}}{\sqrt{1+\kappa}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\kappa}}\right)\right)$$

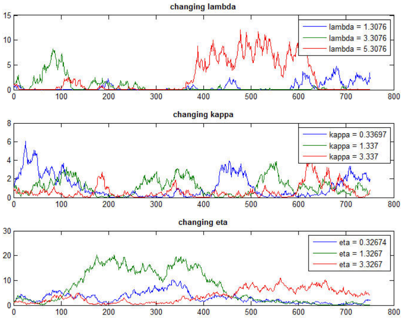
$$\kappa = -\frac{2iu}{\lambda b^2}$$

Example - VG-GOU and VG-CIR

Gamma Ornstein-Uhlenbeck



CIR



Models for CPPI Strategies

Extend the Models to cover a bright range of asset classes
Since there are many asset classes involved the Gaussian hypothesis is too restrictive

- Use complex processes (e.g. NIG (Normal Inverse Gaussian) or VG (Variance Gamma)) Compute the efficient frontier
- Optimization is complex

Therefore

- We need a method to compute relevant figures from time series data
- We need a method to compute the efficient frontier
- We need a method to simulate fairly complex multidimensional processes for creating optimization data

Usage of CPPI with Advanced Models I

For simulating the CPPI strategy

- Time Series Analysis
- Methods to determine figures from given historic data
- Optimization
- What is the best suited characterisation of risk?
- Simulation
- Flexible, robust Monte Carlo Engine

Usage of CPPI with Advanced Models II

To use more complex stochastic processes we must be able to extract the relevant data to determine the processes parameters from market data.
For Geometric Brownian motion this can be done by computing the mean and the covariance structure using time series data.
Therefore, we have to investigate for methods to compute the necessary parameters. We used a version of the *Expected Maximum Likelihood Method* to obtain the parameters. The basic method is described in McNeil and Embrechts (2005)

Generalized Hyperbolic

The one dimensional density for a generalized hyperbolic distribution is given by:

$$f(x,\lambda,\xi,\psi,\mu,\sigma,\gamma) = \frac{\sqrt{\psi\xi}^{-\lambda}\psi^{\lambda}\left(\psi+\frac{\gamma^2}{\sigma^2}\right)^{0.5-\lambda}}{\sqrt{2\pi}\sigma K_{\lambda}(\sqrt{\psi\xi})} \frac{K_{\lambda-0.5}\left(\sqrt{\left(\xi+\frac{(x-\mu)^2}{\sigma^2}\right)\left(\psi+\frac{\gamma^2}{\sigma^2}\right)}\right)e^{\gamma\frac{(x-\mu)^2}{\sigma^2}}}{\sqrt{\left(\xi+\frac{(x-\mu)^2}{\sigma^2}\right)\left(\psi+\frac{\gamma^2}{\sigma^2}\right)}^{0.5-\lambda}} \quad (1)$$

$$\begin{array}{ll} \xi > 0, \psi \geq 0 & \text{if } \lambda < 0 \\ \xi > 0, \psi > 0 & \text{if } \lambda = 0 \\ \xi \geq 0, \psi > 0 & \text{if } \lambda > 0 \end{array}$$

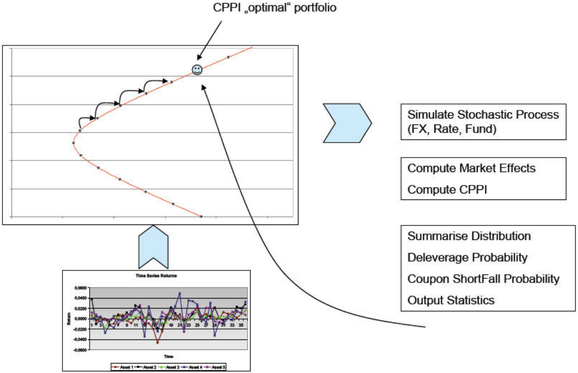
The Parameters

The parameters ξ, ψ and λ determine the distribution and the tail behaviour.

For example $\lambda = 0.5$ corresponds to the NIG model or $\lambda = -\nu/2$, $\xi = \nu$ and $\psi = 0$.

The parameters γ, μ and σ are scale parameters.

Concept Map for Basket Selection



How to obtain the Efficient Frontier?

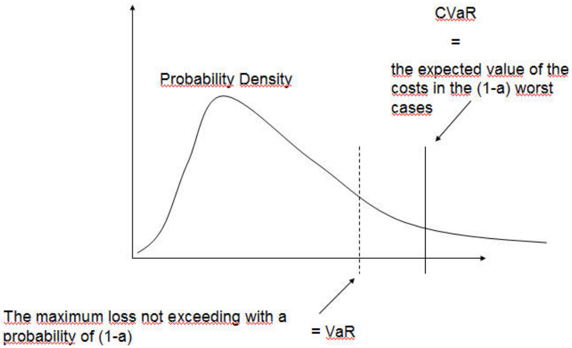
The classical Markowitz/CAPM approach assumes normality. The volatility (in fact VaR) is the measure of risk in this case.

Mostly, computing of VaR relies on linear approximations of portfolios and/or assume a multi-variate normal distribution.

Urysaev and Rockafellar describe a method based on simulated scenarios to compute the VaR, resp. CVaR.

To this end complex multidimensional distributions (skewed, fat tailed, etc.), copula approaches or stochastic volatility with jumps models can be used for asset allocation.

CVaR Illustration



Defining Suitable Representation of CVaR

We assume that the vector $\underline{x}$ denotes the weight vector describing the weight of the assets in the portfolio, $\underline{y}$.
The loss distribution density is given by $l(x, y)$.
The loss of the portfolio not exceeding α is given by:

$$\mathrm{PL}_\alpha(x) = \int_{l(x,y)\leq\alpha} p(y)dy \tag{2}$$

Definition of VaR and CVaR

The β-*VaR* is given by:

$$\alpha_\beta(\underline{x}) \min\{\alpha \in \mathbb{R} : \mathrm{PL}_\alpha(\underline{x}) \geq \beta\} \tag{3}$$

The β-*CVaR* is given by:

$$(1-\beta)^{-1} \int_{l(\underline{x},\underline{y}) \leq \alpha_\beta(\underline{x})} l(\underline{x},\underline{y}) p(\underline{y}) dy \tag{4}$$

Representation of CVaR used for Optimization

$$F_{alpha,\beta}(\underline{x}) = \alpha + (1-\beta)^{-1} \int_{y \in \mathbb{R}^d} [f(\underline{x}, \underline{y}) - \alpha]^+ p(\underline{y}) dy \qquad (5)$$

This function is convex and continuously differentiable and

$$\beta - \mathrm{CVaR} = \min_{\alpha \in \mathbb{R}} F_{\alpha,\beta}(\underline{x}) \qquad (6)$$

and denoting the interval of values of α where the minimum is contained by $A_\beta(x) := \mathrm{argmin}_{\alpha \in \mathbb{R}} F_{\alpha,\beta}(x)$ The VaR is the left endpoint of this interval, ie. β-VaR=inf $A_\beta(x)$.

Description of the Numerical Method

We assume now we have simulated $y_1, y_2, \ldots, y_N$ and consider

$$\hat{F}_{\alpha,\beta}(\underline{x}) = \alpha + \frac{1}{N(1-\beta)} \sum_{k=1}^{N} [f(\underline{x}, \underline{y}_k) - \alpha]^+ p(\underline{y}_k)$$

This function is convex and piecewise linear in α but not differentiable. It can be minimized by linear programming methods and therefore we have approached CVaR calculation by using simulation together with linear optimization to compute the portfolio weigts x and minimize the CVaR.

Choosing $l(\cdot,\cdot)$

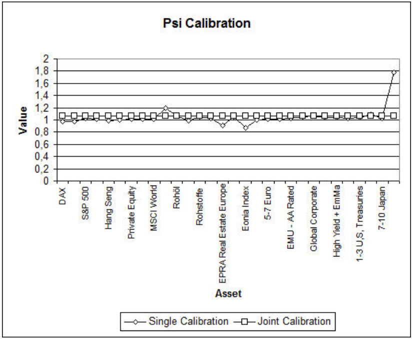
In the simples case (no real constraints) we can apply this approach to

$$l(x,y) = -\underline{x}^T \underline{y}$$

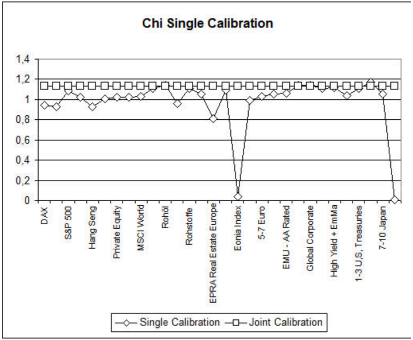
and the function $\hat{F}$ becomes:

$$F-\beta(\underline{x},\alpha) = \alpha + (1-\beta)^{-1} \int_{\underline{y}\in\mathbb{R}^d} \max(-\underline{x}^T \underline{y} - \alpha, 0) p(\underline{y}) dy$$

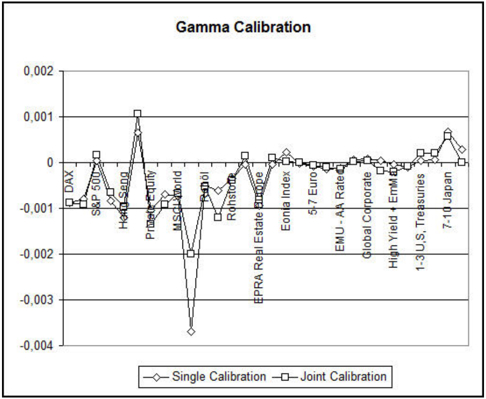
Ψ Calibrated to Market Data



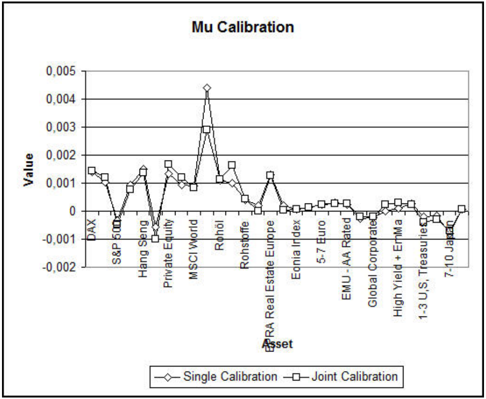
χ Calibrated to Market Data



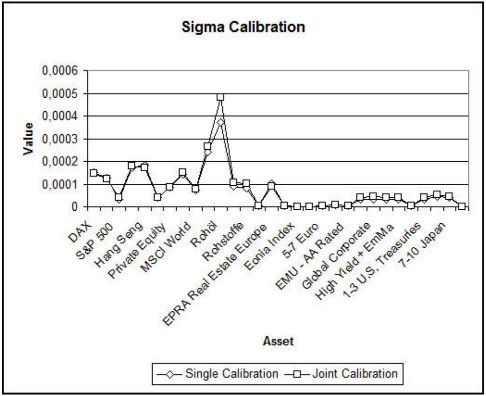
$\underline{\gamma}$ Calibrated to Market Data



μ Calibrated to Market Data



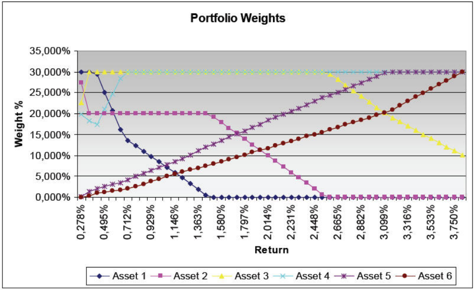
$\underline{\sigma}$ Calibrated to Market Data



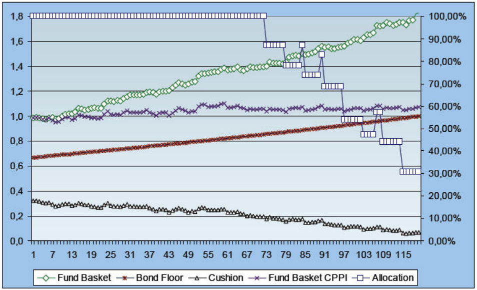
Calibrated Market Data

	DAX	STOXX50	S&P 500	TOPIX	Hang Seng	Hedge Funds 3000	Private Equity	MSCI Asia	MSCI World
Chi =	1,862372867	0	0	0	0	0	0	0	0
Psi =	1,766944094	0	0	0	0	0	0	0	0
Gamma =	-0,000872054	-0,00092565	0,000166394	-0,000649556	-0,000985455	0,001069498	-0,001343017	-0,000927401	-0,00067933
Mue =	0,001434802	0,00121115	-0,000477953	0,000757271	0,001351951	-0,000991789	0,001654638	0,001190945	0,000841121
Sigma =	0,00014842	0,000126663	1,32223E-05	4,1413E-05	5,02147E-05	1,3421E-05	6,58256E-05	4,64862E-05	7,54985E-05
	0,000126663	0,000120919	1,07988E-05	3,74447E-05	4,68693E-05	1,07974E-05	6,29526E-05	4,14661E-05	6,84175E-05
	1,32223E-05	1,07988E-05	3,94985E-05	2,07715E-05	3,60317E-05	3,92139E-05	1,20726E-05	2,77761E-05	3,11579E-05
	4,1413E-05	3,74447E-05	2,07715E-05	0,000181544	8,98096E-05	2,11336E-05	5,96785E-05	0,000154096	5,11504E-05
	5,02147E-05	4,68693E-05	3,60317E-05	8,98096E-05	0,000172978	3,59823E-05	5,13364E-05	0,000111627	6,16173E-05
	1,3421E-05	1,07974E-05	3,92139E-05	2,11336E-05	3,59823E-05	4,15459E-05	1,24876E-05	2,82278E-05	3,11355E-05
	6,58256E-05	6,29526E-05	1,20726E-05	5,96785E-05	5,13364E-05	1,24876E-05	8,60769E-05	5,88556E-05	5,07655E-05
	4,64862E-05	4,14661E-05	2,77761E-05	0,000154096	0,000111627	2,82278E-05	5,88556E-05	0,000151562	6,02142E-05
	7,54985E-05	6,84175E-05	3,11579E-05	5,11504E-05	6,16173E-05	3,11355E-05	5,07655E-05	6,02142E-05	7,60871E-05
	8,24243E-05	8,03428E-05	2,73139E-05	6,19798E-05	9,15908E-05	2,80089E-05	6,32346E-05	7,8814E-05	7,47797E-05
	-5,48197E-06	-1,42713E-07	1,66622E-05	1,17042E-05	2,40932E-05	1,7474E-05	6,52037E-06	2,11035E-05	9,64909E-06
	1,07162E-05	1,18344E-05	-1,72724E-06	1,16177E-05	8,62074E-06	-9,2797E-07	1,29454E-05	1,29374E-05	1,04641E-05
	1,83221E-05	1,81598E-05	2,16254E-05	2,04325E-05	3,18706E-05	2,19239E-05	1,62602E-05	2,72434E-05	2,7497E-05
	-3,59847E-06	-3,41157E-06	-6,11238E-07	-4,02436E-06	-3,19257E-06	-6,32665E-07	-3,5963E-06	-3,14189E-06	-1,94558E-06
	5,5816E-05	5,57154E-05	4,2012E-06	2,07651E-05	3,29969E-05	4,61337E-06	4,75395E-05	2,54136E-05	3,82546E-05

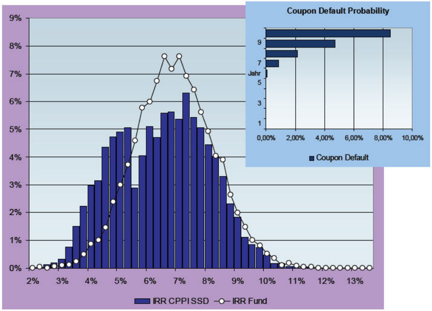
Portfolio Weights



Typical Path



Distribution



Pricing

We observe that models which are calibrated to the same set of options produce different prices for exotic options (see Albrecher, Mayer, and Tistaert (2006)). This is due to the fact that

- we only fix the marginal distributions for the option maturities of a given calibration set
- we use a certain distance measure for calibration
- the models imply different forward volatility surfaces
- the models show a significant different sample path behaviour
- we have to choose a martingale measure

Martingale Measures

The considered process (S or X) is not a martingale in general. To apply arbitrage free pricing we have to make the processes into a martingale!

Working in complete markets such as it is assumed for the Black-Scholes-Merton model there is exactly one possibility to make the process into a martingale.

For general models there are many possibilities:

- Stochastic / Ordinary Exponentials
- Mean Correction MM
- Esscher Transform MM
- Minimal Entropy MM

Pricing Simple Options

For a given index, plain vanilla options such as European calls and puts are liquid instruments.

A given model should price such instruments consistent with the market.

The pricing of such options must be fast. We prefer:

- Analytic Solutions
- Semi-Analytic Solutions

Numerical Methods

In general we have to apply numerical methods to perform pricing, hedging and risk management in our models.

The main techniques we apply are:

1. Fourier Transform Methods / Integration Methods
2. Monte Carlo Methods

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Fourier Transform Methods

In our setup $S(t) = S(0)\exp(L(t))$, the price of a European option with payoff h is given by:

$$V_h(T,K) = \frac{e^{-Rs}}{2\pi}\int_{\mathbb{R}} e^{ius}\varphi(-u-iR)\hat{h}(u+iR)du$$

with φ and $\hat{h}$ being the characteristic functions of the process L and payoff h, respectively. The simplest case of call and put options: $\hat{h}(u) = \frac{K^{1+iu}}{iu(1+iu)}$.

Improvements

The method has been introduced in Carr and Madan (1999) and has been improved in several ways:

1. In terms of speed and accuracy
   - Cosine Method, Fang and Osterlee (2008)
   - Black-Scholes adjustment, Cont and Tankov (2004)
   - Generalized Fourier transform, Lewis (2001)
2. In terms of option payoffs (forward starts, digitals, gap options, options on minimum or maximum)
   - Generalized Fourier transform, Lewis (2001)
   - Forward Start Options, Beyer and Kienitz (2009)
   - Wiener-Hopf Methods, Eberlein, Glau, and Papapantoleon (2008)
3. In terms of early exercise opportunities
   - CONV method, Lord (2008)

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Fourier Transform Methods - Plain Vanilla

Speed

Since we want to use the methods for calibration to liquid market instruments speed is very important!
Plain Vanilla options can be priced very efficiently. For example an option surface consisting of 121 (11 strikes, 11 maturities) can be priced within a fraction of a second.

First, the algorithm produces prices along the whole range of strikes for a given maturity.
Second, parallelize the computation by computing the range of maturities on different cores.

Fourier Transform Methods - Exotics

Prices for some exotic options can be computed using Fourier Transform methods.

Forward Start Options

Beyer and Kienitz (2009) consider Forward Start options. They give an explicit expression of the *Forward Characteristic Function* for a variety of models which can be put into the pricing algorithm.

Min/Max

Eberlein, Glau, and Papapantoleon (2008) consider options including the Min and/or the Max of an asset.

Forward Starting Options

For models based on Lévy processes Beyer and Kienitz (2009) show that the forward characteristic function is given by:

$$\begin{aligned} f_1(u) &= f_2(-\mathrm{i}\psi_L(u))\mathbb{E}\left[\exp(\mathrm{i}f_3(-\mathrm{i}\psi_L(u))y_{t^*})\right] \\ &= f_2(-\mathrm{i}\psi_L(u))\phi_{y_{t^*}}(f_3(-\mathrm{i}\psi_L(u))) \end{aligned}$$

with $\Delta^* = T - t^*$ and

$$\begin{aligned} f_2(u) &= \frac{\exp(\kappa^2\eta\Delta^*/\lambda^2)}{(\cosh(\gamma(u)\Delta^*/2) + \kappa\sinh(\gamma(u)\Delta^*/2)/\gamma(u))^{2\kappa\eta/\lambda^2}} \\ f_3(u) &= \frac{2u}{\kappa + \gamma(u)\coth(\gamma(u)\Delta^*/2)}. \end{aligned}$$

Monte Carlo Methods

We consider an asset $S(t)$, $t \in [0, T]$. To apply the Monte Carlo method to price an option with payoff $h : \mathbb{R}^N \to \mathbb{R}$ which depends on the value of the asset at time steps $t_1, \ldots, t_N = T$ we have to generate a set of $NSim$ discrete sample paths $\hat{S}^k(t_1), \ldots, \hat{S}^k(t_N)$ and evaluate h on these paths.
We denote the value for path k by $\hat{V}^k$. The Monte Carlo estimator is then given by:

$$\hat{V} = \frac{1}{NSim} \sum_{k=1}^{NSim} \hat{V}^k.$$

For details and improvements on the basic Monte Carlo method see Duffy and Kienitz (2009) or Glasserman (2004).

Monte Carlo Methods - Diffusion / Jump Diffusion and Stochastic Volatility Models

Simulating diffusion and jump-diffusion models has long be studied and is now a well-known task. See also Duffy and Kienitz (2009) or Glasserman (2004).

However, different models need a special purpose simulation scheme, for example the Heston model.
General numerical recipes like simply using an Euler discretisation is not recommended!

Monte Carlo Methods - Lévy Models

Several methods for simulating paths for models based on Lévy processes have been suggested.

- Time changing a Brownian Motion
- Direct simulation
- Using background driving Lévy processes
- Approximation using Jump-Diffusion processes

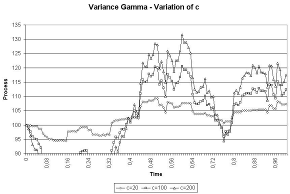
Monte Carlo Methods - VG Model

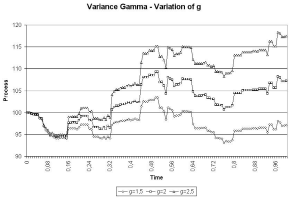
We consider the Variance Gamma model since we base our numerical examples on it.

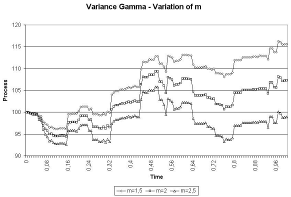
- Difference of two Gamma random variables (C, G, M) representation
  $U \sim \Gamma(\Delta \cdot C, M)$, $D \sim \Gamma(\Delta \cdot C, G)$
  $X(t + \Delta) = X(t) + (r - d) \cdot \Delta + U - D$
- Time changing a Brownian motion (μ, ν, σ) representation
  $h \sim \Gamma(\Delta/\nu) \cdot \nu$, $N \sim \mathcal{N}(0, 1)$
  $X(t + \Delta) = (r - d)\Delta + \mu h + \sigma\sqrt{h}N$

Example - VG

Let us consider the effect of changing the model parameters.







Monte Carlo Methods - Stochastic Volatility Lévy Models

We consider Lévy model with a stochastic clock. To simulate we first simulate the business time and use the integrated business time to simulate the stock price

1. Simulate $y(t), t \in 0, T$
2. Compute $Y(t) = \int_0^t y(s)ds$ using numerical integration
3. Simulate $X(Y(t)), t \in [0, T]$ with respect to the chosen measure

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Monte Carlo Methods - VG GOU/CIR Model

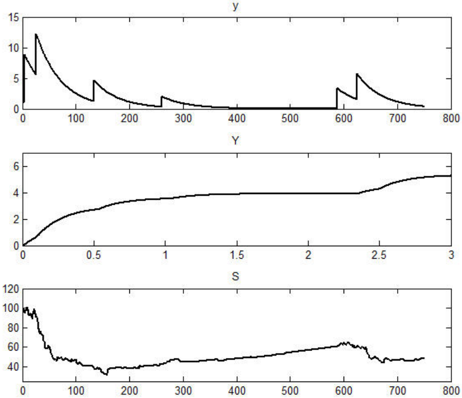
Simulating a Gamma Ornstein-Uhlenbeck process can be accomplished by the following scheme:

- Generate jump times $N \sim \mathcal{P}(a\lambda \cdot \Delta)$
- For $i = 1, \ldots, N$, generate $u_i \sim \mathcal{U}(0,1)$, $v_i = \ln(u_i)/b$
- For $i = 1, \ldots, N$, generate $u_i \sim \mathcal{U}(0,1)$
- $y(t+\Delta) = (1 - \lambda \cdot \Delta)y(t) + \sum_{i=1}^{N} v_i \cdot \exp(-u_i \lambda * \Delta)$

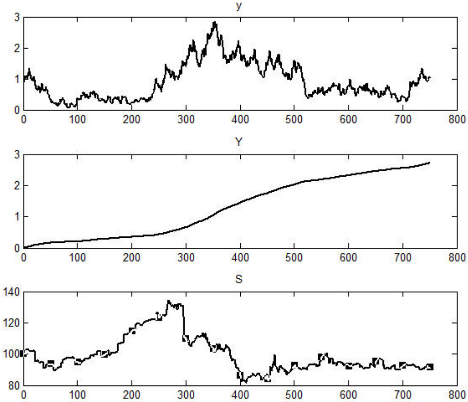
Simulating a CIR process is a well known task! (see for example Andersen (2006))

Monte Carlo Methods - Numerical Example

Gamma Ornstein-Uhlenbeck



CIR



Calibration and Constraints

We have to infer the model parameters using the available market data under given model constraints.

Calibration

- Specify Penalty
- Solve a (constrained) optimization problem
- Statistical methods such as time-series analysis

Constraints

- Simple Constraints (lower / upper bounds on model parameters)
- Linear Constraints
- Non-Linear Contraints

Calibration

Assume N option prices are given and the model depends on the parameter set Θ. We consider weights $\omega_k, k = 1, \ldots, N$ and a measure $\mathbb{Q}$ measuring the distance of the market and the model prices.

The *objective function* is given by:

Objective Function

$$OF(\Theta) = \sum_{k=1}^{N} \omega_k \mathbb{Q}(P^{\Theta}_{\text{Model}}, P_{\text{Market}})$$

Some typical objective functions

The following objective functions are commonly applied:

- root mean square: $\frac{1}{\sqrt{N}}\sqrt{\sum_{k=1}^{N}(P_{\text{Model}}^{\Theta}-P_{\text{Market}})^2}$
- average absolute error: $\frac{1}{N}\sum_{k=1}^{N}\left|P_{\text{Model}}^{\Theta}-P_{\text{Market}}\right|$
- absolute absolute error as percentage of the mean price: $\frac{N}{\sum_{k=1}^{N}P_{\text{Market}}}\sum_{k=1}^{N}\frac{|P_{\text{Model}}^{\Theta}-P_{\text{Market}}|}{N}$
- average relative percentage error: $\frac{1}{N}\sum_{k=1}^{N}\frac{|P_{\text{Model}}^{\Theta}-P_{\text{Market}}|}{P_{\text{Market}}}$

In practice we apply weighting of each summand!

Optimization

(I) Local Optimisation
- LBFGSB
- SQP
- Levenberg-Marquart

(II) Global Optimisation
- Differential Evolution
- Simulated Annealing
- Random Local Search

Stability

Small changes in the risk factors should only apply small changes in the calibrated model parameters!

- Parsimonious structures
- Degrees of freedom
- Model choice

Hedging

A hedge is a trading strategy to minimize risk of a derivative or a portfolio. Several well known methods have been suggested:

- Static Replication
- Dynamic Replication
  - Delta Hedging
  - Delta-Gamma Hedging
  - ...
- Superhedging
- Quadratic Hedging Methods
  - local quadratic hedging
  - global quadratic hedging - variance optimal hedge

Hedging - Good Hedges / Bad Hedges

Hedging is as important as pricing since both concepts are linked!
We can classify a hedge in terms of characteristics of its P&L-distribution.
A good hedge needs:

1. Mean $= 0$
2. Small Variance
3. Small skew and kurtosis

Further questions arise about the tails of the distribution, discrete hedging issues, feasible hedges and numerical considerations for simulating a hedge analysis.

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Further questions arise about the tails of the distribution, discrete hedging issues, feasible hedges and numerical considerations for simulating a hedge analysis.

Hedging - Sensitivities

All approaches to hedging the changes in option price (V) of an (exotic) option due to model parameters such as S_0, implied volatility or volatility skew have to be computed.

Most important are:

- Delta: $\Delta = \frac{\partial V}{\partial S_0}$
- Gamma: $\Gamma = \frac{\partial^2 V}{\partial S_0^2}$
- Vega: $\mathcal{V} = \frac{\partial V}{\partial \sigma}$

For exotic options such sensitivities have to be computed using Monte Carlo methods. To this end we apply the methods in Kienitz (2008a) and Kienitz (2008b).

Greeks using Monte Carlo Methods

The simplest method, so in general not the one suited for Monte Carlo, is the *Finite Difference Method*:

$$\frac{\partial}{\partial \Phi}\mathbb{E}\left[h(S^{\Phi})\right] \overset{\text{FD}}{\approx} \frac{1}{2\epsilon}\left(\mathbb{E}[h(S^{\Phi+\epsilon})] - \mathbb{E}[h(S^{\Phi-\epsilon})]\right)$$

$$\overset{\text{MC}}{\approx} \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2\epsilon}\left(h(\hat{S}_i^{\Phi+\epsilon}) - h(\hat{S}_i^{\Phi-\epsilon})\right)$$

Other methods like the *Pathwise Method* (PW) or the *Likelihood Ratio Method* (LRM) have been invented. In principle, the LRM can be applied to our problem but the mathematical derivations become very complicated! We will rely on a method based on *Importance Sampling* introduced for computing Greeks in Libor Market Models, see Fries (2007) or Fries and Kampen (2005).

The Full Proxy Method

We denote the number of simulations by $NSim$ and the number of time steps per simulation by $NTime$. We apply the following algorithm:

- For each run $i = 1, \ldots, \mathrm{NSim}$ simulate $\hat{S}^i_j$ at $t_j \in \{t_1, \ldots, t_{\mathrm{NTime}} = T\}$
- Compute the weights $\omega^{\mathrm{path}}_{i,j}$
- Compute the weights $\omega_i = \prod_{j=1}^{\mathrm{NTime}} \omega^{\mathrm{path}}_{i,j}$
- Compute the payoff $\hat{V} = \frac{1}{\mathrm{NSim}} \sum_{i=1}^{\mathrm{NSim}} h(\hat{S}_i)\omega_i(\hat{S}_i)$

We used the fact that we can compute the weights ω_i by:

$$\omega_i(y) = \prod_{j=1}^{\mathrm{NTime}} \frac{f^{\Phi+\epsilon}(y) - f^{\Phi-\epsilon}(y)}{f^{\Phi}(y)} = \prod_{j=1}^{\mathrm{NTime}} \omega^{\Phi+\epsilon} - \omega^{\Phi-\epsilon}$$

Numerical Example - Digital Option

We consider the Merton model and a digital option. We use the parameters

- $S(0) = 100,\ K = 100,\ T = 1.0,$
- $r = 0.07,\ d = 0.0,\ \sigma = 0.2,$
- $\lambda = 0.5,\ \mu_J = 0.05,\ \sigma_J = 0.15$

and a disturbance of the inital parameters by 1% and the discretization $\tau = \{0, T\}$.

Method	PV (SE)	Δ (SE)	Γ (SE)	v (SE)
Analytic	0.531270	0.016610	-2.800324e-004	-0.560070
Monte Carlo Proxy	0.531576 (0.001460)	0.016641 $(7.195974e-005)$	$-2.844054e-004$ $(4.769032e-006)$	-0.569254 (0.009541)
Monte Carlo FD	0.531576 (0.001460)	0.016485 $(2.722766e-004)$	-0.0428901 (0.018318)	-0.0022114 (0..12765)

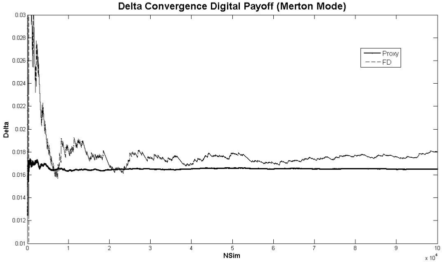
Applying the Proxy Method

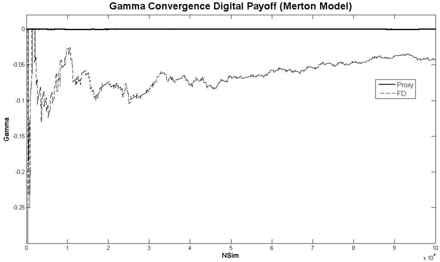
We need the transition density for the Merton model. It is given by:

$$f_M(x, t+\Delta; x_t, t) = \sum_{i=0}^{\infty} \frac{e^{-\lambda\Delta}(\lambda\Delta)^i}{\sqrt{2\pi}\sqrt{i\sigma_J + \Delta\sigma^2} i!} \exp\left(-\frac{(z(x) - i\mu_J)^2}{2(i\sigma_J + \Delta\sigma^2)}\right)$$

$$z(x) = x - x_t - (r - d + \tfrac{\sigma^2}{2})\Delta + \lambda\left(\exp\left(\mu_J + \tfrac{\sigma_J^2}{2}\right) - 1\right)$$

Greeks - Merton Model and Digital Option





Numerical Example - Knock Out Option

We consider the Merton model and a digital option. We use the parameters

- $S(0) = 100,\ K = 100,\ T = 1.0,$
- $r = 0.07,\ d = 0.0,\ \sigma = 0.2$
- $\lambda = 0.5,\ \mu_J = 0.05,\ \sigma_J = 0.15$

We consider a knock-out option with time-dependent barriers. Such options typically have to be priced using Monte Carlo simulation. For the numerical studies we use:

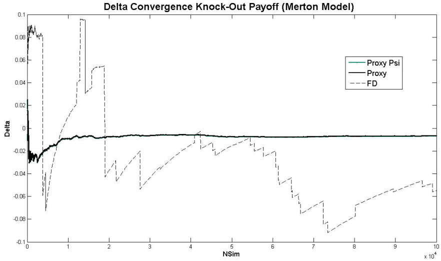
$$L(t) = \begin{cases} 90 & \text{if} \quad t \in (0, 0.5] \\ 85 & \text{if} \quad t \in (0.5, 0.75] \\ 80 & \text{if} \quad t \in (0.75, 1] \end{cases} \quad U(t) = \begin{cases} 110 & \text{if} \quad t \in (0, 0.5] \\ 115 & \text{if} \quad t \in (0.5, 0.75] \\ 120 & \text{if} \quad t \in (0.75, 1] \end{cases}$$

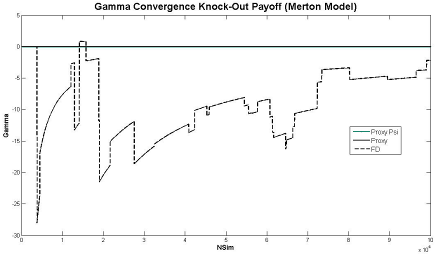
The Knock-Out Option

The numerical results are:

Method	PV (SE)	Δ (SE)	Γ (SE)	ν (SE)
Monte Carlo	0.564746 (0.0075695)	-0.0066871 $(0.9910932e-004)$	-0.0072713 $(1.8442553e-004)$	-8.6071757 (0.1745823)
Monte Carlo FD	0.564746 (0.0075695)	-0.054896 (0.001825)	-2.167931 (6.893258)	-0.0174076 (4.8101873)

Greeks - Merton Model and Knock Out Option





Numerical Example

We consider the Variance Gamma model and a digital option. The parameters are:

- $S(0) = 100,\ K = 100,\ T = 1.0,\ r = 0.1,$
- $\sigma = 0.12136,\ \mu = -0.1436$ and $\nu = 0.1686$

The parameters have been chosen according to Madan (1998) or Ribeiro and Webber (2007).

Method	PV (SE)	Δ (SE)	Γ (SE)	ν (SE)
Analytic	0.697421	0.01844	-0.00136	-1.137605
Monte Carlo	0.69721 $(8.5077e-002)$	0.01841 $(1.2547e-004)$	-0.00134 (3.707097)	-1.371968 (0.01568)
Monte Carlo FD	0.69721 $(8.5077e-002)$	0.0189337 $(2.6670e-004)$	$-4.5241871e-004$ (0.083867)	-1.658869 (0.35365)

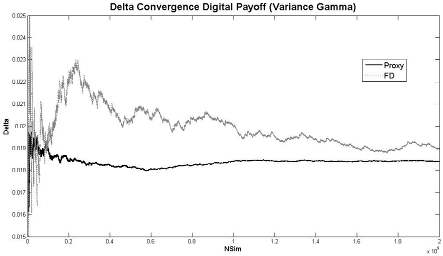
Greeks - Variance Gamma Model

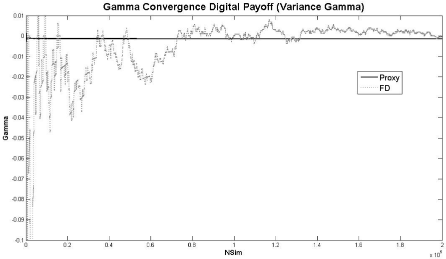
We need the transition density for the model. It is given by:

$$f_t^{VG}(x) = \frac{2\exp\left(\frac{\mu x}{\sigma^2}\right)}{\nu^{t/\nu}\sqrt{2\pi}\sigma\Gamma(t/\nu)} \left(\frac{x^2}{\frac{2\sigma^2}{\nu}+\mu^2}\right)^{t/2\nu-1/4} K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{x^2\left(\frac{2\sigma^2}{\nu}+\mu^2\right)}\right)$$

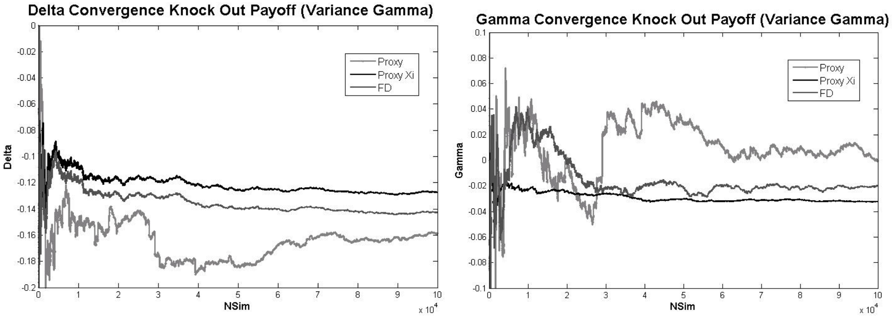
$z(x) = x - x_t - (r+\omega)\Delta_t$ and
$K_\nu(z) = \frac{1}{2}\int_0^\infty y^{\nu-1}\exp\left(-\frac{z}{2}(y+y^{-1}\right) dy$ denotes the modified Bessel function of the second kind

Greeks - Variance Gamma Model and Digital Option





Greeks - Variance Gamma Model and Knock Out Option



Greeks - VG - GOU Model

For this model we do not have an analytical expression for the density. Therefore, we have to rely on numerical approximations. The characteristic function and the probability density of random variable Z are related by:

$$\hat{f}(u) = \int_S \exp(iux) f(x) dx; \quad f(x) = \frac{1}{2\pi} \int_S \exp(-iux) \hat{f}(u) fu$$

we consider f on the discrete set

$$u_k = \pi / A \left(k - N/2 \right), k = 0, \ldots, N-1$$

which leads to:

$$\hat{f}(u_k) = \int_S \exp(iu_k x) f(x) dx \approx \Delta x \sum_{j=0}^{N-1} \exp(iu_k x_j) f(x_j) \quad k = 0, \ldots, N-1$$

Using conjugate symmetry we find:

$$
\begin{aligned}
&\exp(-iu_k x_j)\phi_t(u_k) + \exp(-iu_{N-k} x_j)\phi_t(u_{N-k}) \\
&= \exp(-iu_k x_j)\phi_t(u_k) + \exp(iu_k x_j)\overline{\phi(u_k)} \\
&= \exp(-iu_k x_j)\phi_t(u_k) + \overline{\exp(-iu_k x_j)\phi_t(u_k)} \\
&= 2\mathcal{R}\left(\exp(-iu_k x_j)\phi_t(u_k)\right)
\end{aligned}
$$

Numeric Method for Probability Density

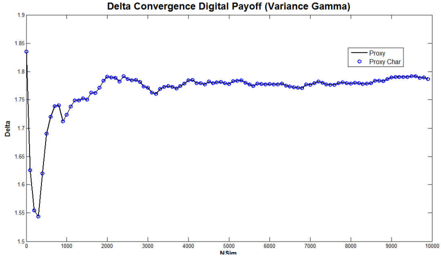
Consider the interval $S = [-A, A]$. We assume that the interval is chosen such that $d(-A) = d(A)$. Let M be a positive integer. To determine the value of the probability density at x we proceed by:

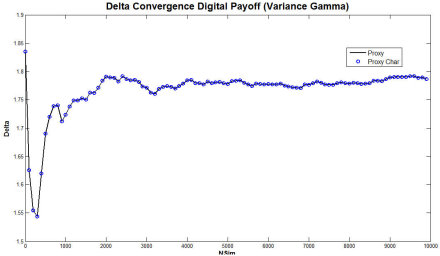
- Set $N = 2^M$
- $u_j = j\pi(1/A - N/(2A)), \quad j = 0, \dots, N-1$
- Set $\delta_u = \pi/A$
- return $\sum_{j=0}^{N-1} 2(\mathcal{R}(\exp(-iu_j x) f(u_j)))\Delta_u/(4\pi)$;

We denote by $\mathcal{R}(\cdot)$ the real part of a complex number.
We remark that the interval S need not be symmetric. We have assumed symmetry just for convenience.

Testing the Approach

We consider a Digital Option and use the analytic density as well as the density computed using Fourier inversion on two parameter sets.



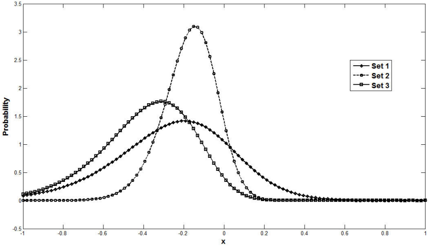


Test Sets

We consider the Variance Gamma model with Gamma Ornstein-Uhlenbeck clock. We tested our methods using the following parameter sets:

Set	C	G	M	λ	a	b
Set 1	6.161	9.6443	16.026	1.679	0.3484	0.7664
Set 2	8.88	24.95	48.19	3.3	0.715	1.031
Set 3	6.47	11.10	33.41	0.94	0.63	1.47

As a benchmark model we consider the VG model where analytical result exist.



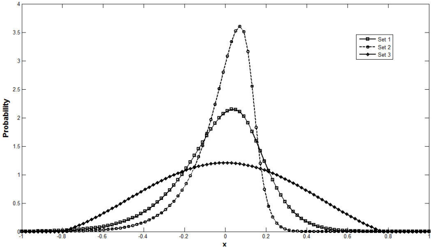


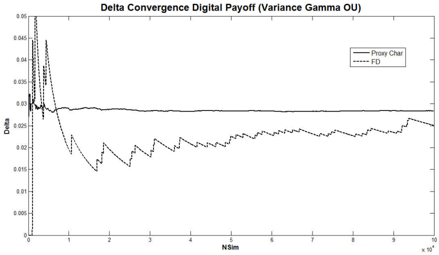
Figure: VG and VGOU densities for test sets 1 - 3

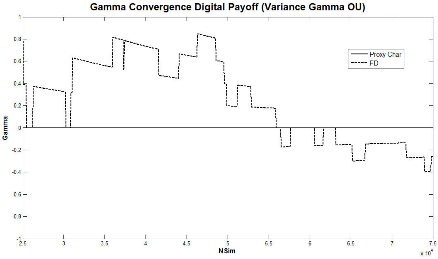
Numerical Example - Digital Option

We consider the Variance Gamma model with Gamma Ornstein-Uhlenbeck clock.
As the payoff we choose a digital option and take Set 2 as the model parameters.
The results are:

Parameter	Proxy	Proxy Char
	(Digital Option)	
Δ_A	0.0275	
Δ	0.0283 (2.4050e-004)	0.0250 (0.0490)
Γ_A	-0.0017	
Γ	-0.0018(4.8989e-005)	-0.001280(0.06930)

Greeks - VG-GOU Model and Digital Option



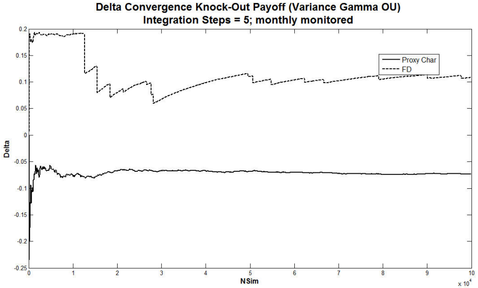


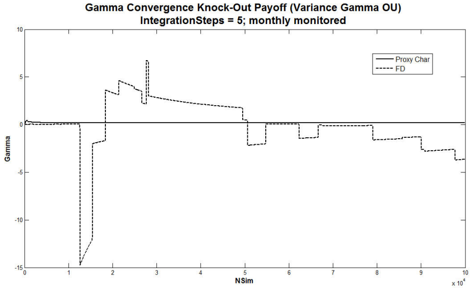
Numerical Example - Knock-Out Option

We consider the Variance Gamma model with Gamma Ornstein-Uhlenbeck clock.
As the payoff we choose now a knock-out option and take again Set 2 as the model parameters.
The results are:

	(Knock-Out Option)	
Parameter	5 Steps per month	10 Steps per month
Δ_{FD}	0.1084	0.1417
Δ	-0.0734(0.0041)	-0.0688 (0.0041)
Γ_{FD}	-3.6413	2.6507
Γ	0.2100(0.0092)	0.2055(0.0098)

Greeks - VG-GOU Model and Knock Out Option





Implementation of the Approach

We have implemented the algorithms to calculate the sensitivities and the pricing of forward start options using Matlab, VBA and C++.

The implementation using VBA is the most time consuming since we had to create and test proprietary implementation of FFT, Bessel functions, etc.

Matlab and C++ (using boost) offer a wide range of fast and tested libraries for performing all the necessary calculations. Furthermore, in terms of speed Matlab as well as C++ are superior to VBA.

Conclusion

Risk management of exotic options ...

- needs sophisticated numerical methods
- needs appropriate modeling
- pricing and hedging are model dependent
- dependent on calibration measures and calibration data
- needs proper implementation

Finally, many thanks to the Deutsche Postbank's Quantitative Analysis team. Especially, Daniel Wetterau for fruitful discussions.

AI - Hedging - Measuring Hedge Performance

Galtschuk-Kunita-Watanabe-Decomposition

$$H = V_0 + \int_0^T \varphi dS(t) + R_T$$

For Lévy:

$$\frac{\sigma^2 \frac{\partial V_t}{\partial S(t)} + \frac{1}{S(t)} \int_{\mathbb{R}} \nu_L(dz)\,(e^z - 1)\,(V_t(S(t)e^x) - V_t(S(t))}{\sigma^2 + \int_{\mathbb{R}} (e^x - 1)\,\nu_L(dz)}$$

For Stochastic Volatility:

$$\frac{\sigma^2 \frac{\partial V_t}{\partial S(t)} + \frac{\partial V(t)}{\partial y(t)} \frac{\rho\xi}{\sigma S(t)} + \frac{1}{S(t)} \int_{\mathbb{R}} \nu_L(dz)\,(e^z - 1)\,(V_t(S(t)e^x) - V_t(S(t))}{\sigma^2 + \int_{\mathbb{R}} (e^x - 1)\,\nu_L(dz)}$$

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