Hedging forward positions: Basis risk versus liquidity costs

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Companies operating a gas power plant have an immanent

- short forward position of natural gas (NG),
- long forward position of power.

To reduce price risk they

- buy natural gas on forward markets,
- sell power on forward markets.

Suppose that a German energy company wants to buy today the NG it needs in 2014.

Problem: German gas forward market is very illiquid.
Bid-ask-spread ↓ as time to delivery approaches

Dutch and German gas prices are **highly** correlated
2 Ways of Hedging

▶ Hedge 1:
Buy natural gas in G

▶ Hedge 2:
Buy natural gas in NL.
Shortly before delivery: sell in NL and buy in G.

Pros & Cons:

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Trade-off: High liquidity costs versus basis risk

Question: What is the optimal position in German and Dutch NG at any time before 2014?

⇒ A singular (stochastic) control problem
The model

- initial short position: $x_0 < 0$
- $T =$ time horizon
- $X_t =$ primary asset position (e.g. German NG);
  
  Constraints: $X_{0^-} = x_0$ and $X_T = 0$

- $Y_t =$ proxy position (e.g. Dutch NG)
  
  Constraints: $Y_{0^-} = 0$ and $Y_T = 0$
Minimizing overall costs $\Leftrightarrow$ minimizing execution costs

- $P_t =$ forward price of the primary asset at time $t$ (a continuous martingale)
- $K_t =$ liquidity costs of primary asset at time $t$ (a non-negative process with cadlag paths)
- $L =$ half bid-ask-spread of proxy

**Expected costs in the primary asset:**

$$E \left[ \int_{[0,T]} P_s dX_s + \int_{[0,T]} K_s |dX_s| \right] = -P_0 x_0 + E \left[ \int_{[0,T]} K_s |dX_s| \right].$$

**Expected costs in the proxy:**

$$E \left[ \int_{[0,T]} L |dY_s| \right]$$

**Expected execution costs**

$$C(X, Y) = E \left[ \int_{[0,T]} K_s |dX_s| + \int_{[0,T]} L |dY_s| \right]$$
The model cont’d

Risk

- $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$ covariance matrix

- Instantaneous risk at time $t$:

$$f(X_t, Y_t) = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}^T \begin{pmatrix} X_t \\ Y_t \end{pmatrix}$$

- Overall risk:

$$R(X, Y) = \int_0^T \sqrt{f(X_t, Y_t)} dt$$
Target function and minimum variance hedge

Control problem:

\[ C(X, Y) + \lambda R(X, Y) \rightarrow \min! \]

Lemma

Let \( X \) be a given primary position path and assume that \( L = 0 \). Then the optimal cross hedge is given by

\[ Y_t^* = -\frac{\sigma_1}{\sigma_2} X_t. \]

\[ h = \frac{\sigma_1}{\sigma_2} = \text{minimum variance hedge ratio} \]
Optimal position paths are (piecewise) monotone

Proposition

Let \((X^*, Y^*)\) be optimal. Then almost surely

a) \(X^*_t\) is non-decreasing, and

b) there exists a càdlàg, adapted and non-decreasing process \(I\) such that \(Y^*_t = I_t \wedge -hX^*_t\).
Our method for getting explicit solutions

**Assumption A:** The optimal cross hedge $Y(X)$ associated to any $X$ is non-increasing after 0, i.e. of the form

$$Y(X)_t = y \land -\rho \frac{\sigma_1}{\sigma_2} X_t.$$ 

**Iterative Method:**

1. For a given $y \geq 0$ determine the optimal primary position $X = X(y)$. To this end reformulate the problem as a stopping problem.
2. Determine optimal initial cross hedge position $y^*$. 
3. The optimal positions are given by
   $$X_t^* = X_t(y^*) \text{ and } Y_t^* = y^* \land -\rho \frac{\sigma_1}{\sigma_2} X_t^*.$$
The primary position via optimal stopping

For any \( y \geq 0 \) consider the problem

\[
E \left[ \int_{[0,T]} K_s \, dX_s + \int_0^T g(X_s) \, ds \right] \rightarrow \min! \tag{1}
\]

where \( g(x) = \lambda \sqrt{f(x, y \wedge -\rho \frac{\sigma_1}{\sigma_2} x)} \).

**Proposition**

For all \( x \in [x_0, 0] \) let \( \tau(x) \) be the solution of the stopping problem

\[
\inf_{\tau \in [0,T]} E [K_\tau + \tau g'(x)].
\]

Then an optimal primary position \( X \) for (1) is given by

\[
X_t = \inf \{ x \in [x_0, 0] | \tau(x) > t \}.
\]
The primary position via optimal stopping

Right continuous inverse of a position path:

\[ \tau(x) = \inf\{ t \geq 0 | X_t > x \} \]

"First time where the position is \( x \)"
The primary position via optimal stopping

Right continuous inverse of a position path:

$$\tau(x) = \inf\{t \geq 0 | X_t > x\}$$

"First time where the position is > x"
The primary position via optimal stopping

The Change of Variables Formula $\implies$

$$E \left[ \int_{[0, T]} K_s dX_s + \int_0^T g(X_s) ds \right] = \int_{x_0}^0 E \left[ K_{\tau(z)} + \tau(z)g'(z) \right] dz + g(x_0) T$$

marginal cost + marginal risk

$\to$ Minimize marginal costs + marginal risk pointwise
Example: Concave deterministic costs

- Liquidity costs are *deterministic* and decreasing
- Speed of decay is *non-decreasing*
Proposition

Suppose that $K$ is decreasing and concave on $[0, T]$. Then the optimal position strategy is of the form

$$X^*_t = x^* 1_{[0, T)}(t) \quad \text{and} \quad Y^*_t = y^* 1_{[0, T)}(t),$$

with $x^* \leq 0$ and $y^* \geq 0$.

The optimal positions $x^*$ and $y^*$ can be calculated explicitly (tedious!).
Example cont’d: Why they are static

**Assumption:** it is not optimal to close a unit of the primary at $t = 0$, i.e.

Cost saving over $[0, \varepsilon] >$ Additional risk over $[0, \varepsilon]$. 

Since risk $\sim$ time, this implies

Cost saving over $[\varepsilon, 2\varepsilon] >$ Additional risk over $[\varepsilon, 2\varepsilon]$. 

$\implies$ It is not optimal to close at $\varepsilon$.

By *iteration:* it is optimal to close at $T$. 

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**Basis risk versus liquidity costs**
Example cont’d: Decision tree

\[ L \geq \bar{L} \]

- no
  - cross hedge with \( A \) futures

- yes
  - keep primary open

\[ \Delta K \geq \lambda \sigma_1 T \]

- no
  - close primary
  - do not cross hedge

- yes
  - keep primary open
  - do not cross hedge

\[ \bar{L} = \frac{\sigma_2}{2\sigma_1} \left( \Delta K \rho - \sqrt{(1 - \rho^2)(\lambda^2 \sigma_1^2 T^2 - \Delta K^2)^+} \right) \]

\[ A = -\frac{\sigma_1}{\sigma_2} \max \left( 0, \rho - 2L \frac{\sqrt{1 - \rho^2}}{\sqrt{\lambda^2 \sigma_1^2 T^2 - 4L^2}} \right) x_0 \]

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Basis risk versus liquidity costs
Example: Convex deterministic costs

- Liquidity costs are *deterministic* and decreasing
- Speed of decay is decreasing
- $L = 0$
Proposition

Suppose that \( L = 0 \) and that \( K \in C^1 \) is decreasing and convex on \([0, T]\). If \( \lambda \sigma_1 \sqrt{1 - \rho^2} \in [-\dot{K}(T), -\dot{K}(0)] \), then the optimal closing time is given by

\[
t^* = (\dot{K})^{-1}(-\lambda \sigma_1 \sqrt{1 - \rho^2}),
\]

and \( X^* = x_0 1_{[0, t^*]} \) and \( Y^* = -\rho \frac{\sigma_1}{\sigma_2} x_0 1_{[0, t^*]} \) are the optimal position processes.
Example cont’d: Optimal buying time

Optimal turning point:

\[ t^* = (\dot{K})^{-1}(-\lambda \sigma_1 \sqrt{1 - \rho^2}) \]

At \( t^* \): marginal cost saving = marginal add. risk
Example: Active trading kicks in at a random time

- $K$ jumps at a random time $\tilde{\tau}$ from a higher level $K_+$ to a lower level $K_-$.  
- $\tilde{\tau}$ is the first jump time of an inhomogeneous Poisson process with non-decreasing jump intensity.

→ Close positions at time $\tilde{\tau}$: $X_s = Y_s = 0$ for all $s \geq \tilde{\tau}$.  

![Diagram showing liquidation costs over time, with a jump at time $\tilde{\tau}$]
Proposition

Suppose that $K$ jumps from $K_+$ to $K_-$ at time $\tilde{\tau}$. Then the optimal position strategy is of the form

$$X^*_t = x^* 1_{[0,\tilde{\tau})}(t) \text{ and } Y^*_t = y^* 1_{[0,\tilde{\tau})}(t),$$

with $x^* \leq 0$ and $y^* \geq 0$.

The optimal positions $x^*$ and $y^*$ can be calculated explicitly.
Conclusion

▶ When hedging on forward markets one frequently has to choose between liquidity costs and basis risk.

▶ We introduce a singular control model allowing to characterize optimal trade-offs.

▶ Optimal position paths can be obtained by solving families of stopping problems.

▶ Examples with deterministic liquidation costs:
  - If the speed of cost decay is non-decreasing, then it is optimal to trade only at 0 or $T$. We have explicit position formulas.
  - If the speed of cost decay is decreasing, then it is optimal to close the position when the marginal cost saving equals marginal add. risk.
Thank you!